Shape-from-Shading and Viscosity Solutions

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Abstract: This research report presents an approach to the shape from shading problem which is based upon the notion of viscosity solutions to the shading partial differential equation, in effect a Hamilton-Jacobi equation. The power of this approach is twofolds: 1) it allows nonsmooth, i.e. nondifferentiable, solutions which allows to recover objects with sharp troughs and creases and 2) it provides a framework for deriving a numerical scheme for computing approximations on a discrete grid of these solutions as well as for proving its correctness, i.e. the convergence of these approximations to the solution when the grid size vanishes.

Our work extends previous work in the area in three aspects. First, it deals with the case of a general illumination in a simpler and a more general way (since they assume that the solutions are continuously differentiable) than in the work of Dupuis and Oliensis [9]. Second, it allows us to prove the existence and uniqueness of *"continuous"* solutions to the shading equation in a more general setting (general direction of illumination) than in the work of Rouy and Tourin [29], thereby extending the applicability of shape from shading methods to more realistic scenes. Third, it allows us to produce an approximation scheme for computing approximations of the "continuous" solution on a discrete grid as well as a proof of their convergence toward that solution.

This report aims to deepen the notions presented in our article [28]. Here we give the proofs of the theorems proposed in [28] and which don't appear in other references. Finally by this text, we want to popularize the notion of viscosity solutions and make it more intuitive. Also we hope to convince the reader of the usability of these tools.

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Key-words: Shape from Shading, viscosity solutions, existence and uniqueness of a solution, Hamilton-Jacobi equations, dynamic programming principle, approximation and numerical schemes, finite difference, convergence of schemes.

“Shape-from-Shading” et solutions de viscosité

Résumé : Ce rapport de recherche présente une approche du “Shape from Shading” basée sur la notion de solutions de viscosité des équations de Hamilton-Jacobi. Les intérêts de cette approche résident essentiellement en deux points:
1) Elle permet d’obtenir des solutions non lisses (non différentiables) et ainsi de pouvoir considérer des solutions représentées par des surfaces contenant certains types d’arêtes.
2) Elle nous permet aussi de déduire des schémas numériques puis des algorithmes permettant d’approximer les solutions sur des grilles discrètes. Cet outil nous permet enfin de prouver la pertinence des algorithmes déduits en démontrant leur convergence.

Nos travaux étendent les précédents dans le domaine sous trois aspects.
Premièrement ils traitent du cas d’une illumination générale dans un cadre plus simple et plus général que celui présenté par Dupuis et Oliensis [9] (Ils supposent que les solutions sont continûment différentiables).
Deuxièmement, ils nous permettent de prouver l’existence et l’unicité de solutions (seulement) continues à “l’équation d’irradiance” dans un cas plus général que celui traité par Rouy et Tourin [29] (ici la direction de l’illumination est quelconque).
Troisièmement, ces outils nous permettent de déduire des schémas et des algorithmes numériques permettant d’approximer ces solutions de viscosité et dont nous pouvons prouver la convergence.

Ce rapport a essentiellement deux buts:
1) Éclaircir les notions introduites dans l’article Prados-Faugeras-Rouy [28]. En particulier, nous donnons en détail les preuves, absentes dans la littérature, des théorèmes proposés dans cet article.
2) Démocratiser la notion de solutions de viscosité: Par ce texte, nous espérons convaincre le lecteur qu’aujourd’hui, cette notion est accessible à tous et qu’elle est devenue très utilisable en pratique.

Mots-clés : Shape from Shading, solutions de viscosité, existence et unicité de solutions, équations de Hamilton-Jacobi du premier ordre, principe de programmation dynamique, différences finies , convergence de schémas.

Chapter 1

Introduction

1.1 Shape from shading; state of art

Shape from shading has been a central problem in the field of computer vision since the early days. The problem is to compute the three-dimensional shape of a surface from the brightness variations in a black and white image of that surface. The work in our field was

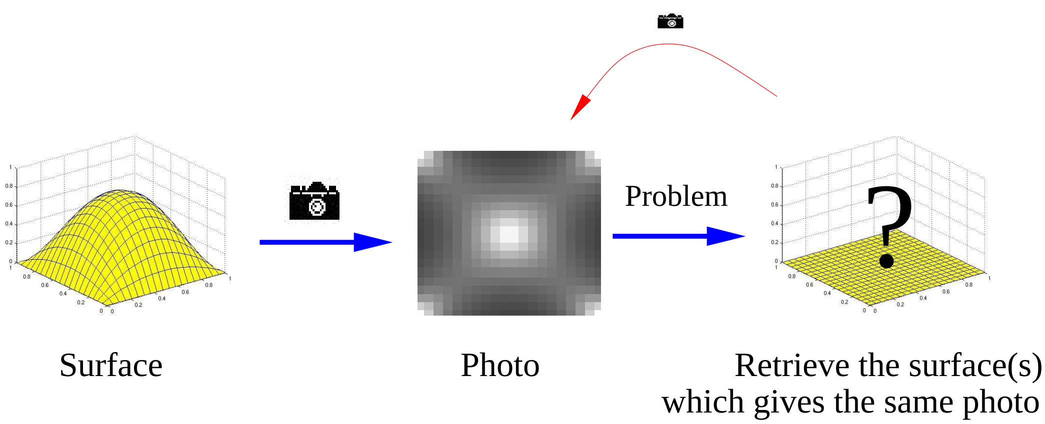


Figure 1.1: The "Shape-from-Shading" problem.

pioneered by Horn who was the first to pose the problem as that of finding the solution of a nonlinear first-order partial differential equation called the brightness equation [18]. This initial idea was limited by the particular numerical method that was used (the method of characteristics) and was enriched by posing the problem as a variational problem [17] within which additional constrains such as those provided by occluding contours [20] can be taken into account. The book [16] contains a very nice panorama of the research in this area up to 1989. Questions about the existence and uniqueness of solutions to the problem were simply not even posed at that time with the important exception of the work of Bruss [5]. These questions as well as those related to the convergence of numerical schemes for computing the solutions became central in the last decade of the 20th century. Brightness equations that do not admit continuously differentiable solution were produced [4, 19], Durou and his co-workers showed that some well-known numerical schemes were in fact almost never convergent [10] and exhibited a continuous family of ambiguous solutions [11]. A breakthrough was achieved by people who realized that control theory could be brought to bear on this problem. Dupuis and Oliensis showed that this theory provided a way of constructing numerical schemes with provable convergence properties in the case where a continuously differentiable solution existed [9]. More significantly perhaps, P.-L. Lions, Rouy and Tourin used the theory of viscosity solution of Hamilton-Jacobi equations to characterize the existence and uniqueness of weak solutions to the brightness equation and to come up with provably convergent numerical schemes to compute them [29, 26]. In doing so, they considerably generalized the applicability of shape from shading since solutions could be only continuous and they opened the way to the mathematically well-posed use of such constraints as occluding edges and shadows as well as general light sources.

1.2 Mathematical formulation

In this report we revisit one of the simplest versions of the shape from shading problem, the idea being that the tools that we develop here will be extendable to more general and realistic situations. We therefore assume that

- the camera performs an orthographic projection of the scene (hence a simple affine camera model as opposed to a pinhole),
- the scene is illuminated by a single point source at infinity; thus, the beam is parallel and we can represent the light by a constant vector.
- that its reflectance is Lambertian and its albedo constant and equal to 1.

We also assume that there are no shadows and no occluding boundaries and that the distance from the camera to the scene is known on the boundary of the image. Admittedly, these hypotheses may appear a bit restrictive. In fact they are not in the sense that they can be generalized without drastically changing the mathematical analysis that is done in this report. But this will be the subject of another report.

We denote by u the distance of the points in the scene to the camera, I the image intensity, $\mathbf{L} = (\alpha, \beta, \gamma)$ the unit vector representing the direction of the light source ($\gamma > 0$), and $\mathbf{l} = (\alpha, \beta)$. The image is modelled as a function from the closure $\overline{\Omega}$ of an open set Ω of $\mathbb{R}^2$ into the closed interval $[0, 1]$, $I : \overline{\Omega} \longrightarrow [0, 1]$. Given our hypotheses, the shape from shading problem is, given I and $\mathbf{L}$, to find a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying the brightness

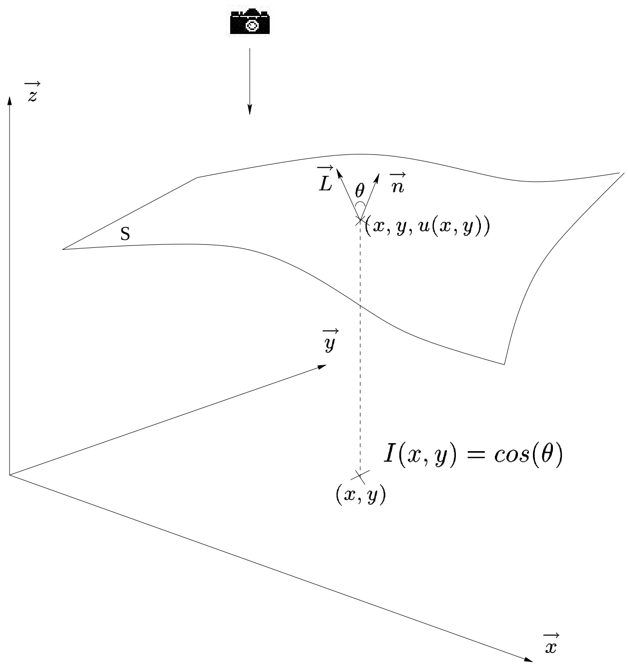


Figure 1.2: Image provided by an orthogonal projection. The intensity of the "pixel"(x, y) is the intensity of the point $(x, y, u(x, y))$ on the surface S.

equation:

$$\forall x \in \Omega, \quad I(x) = \frac{-\overrightarrow{\nabla u}(x) \cdot \mathbf{l} + \gamma}{\sqrt{1 + |\overrightarrow{\nabla u}(x)|^2}}, \tag{1.1}$$

$$\text{with the Dirichlet boundary counditions} \quad \forall x \in \partial\Omega, \quad u(x) = \varphi(x), \tag{1.2}$$

φ being continuous on $\partial\Omega$.

Note that in the case where the light source is in the same direction as the direction of projection (it is the case considered in [29]) we have $\mathbf{L} = (0, 0, 1)$, and the PDE (1.1) is equivalent to an Eikonal equation:

$$|\overrightarrow{\nabla u}(x)| = \sqrt{\frac{1}{I(x)^2} - 1}. \tag{1.3}$$

Note also that (1.1) is a Hamilton-Jacobi equation and can be rewritten as $H(x, \overrightarrow{\nabla u}(x)) = 0$, where H is the Hamiltonian (see section 4.1).

Chapter 2

Viscosity solutions

This chapter presents a brief introduction to the notion of viscosity solutions of first order Hamilton-Jacobi equations. Its aim is first to present the fundamental definitions and theorems which are necessary to understand the results presented in the following chapters. In particular, we will only consider theorems dealing with Dirichlet problems. The second purpose of this section is to trivialize this notion; we hope to reach this goal thanks to various remarks and many cross-references towards figures. Also, we hope that this document is understanding by himself; it is to say, for understand it, the reader shouldn't need all the usual references. Of course, this introduction being very well-targeted, we advise interested readers to rapidly get hold of most of quoted references.

At last, in this section we attempt too to warn the reader about some difficulties and we illustrate them with a classical example.

In a first time, we introduce the continuous viscosity solutions notion. Next we will talk about discontinuous viscosity solutions.

2.1 Continuous viscosity solutions

The notion of viscosity solutions of Hamilton-Jacobi equations has been introduced by Crandall and Lions [6, 25, 8, 7] in the 80s. It is a very nice way of making quantitative and operational the intuitive idea of weak solutions of first-order (and for that matter, second-order) Partial Differential Equations (PDEs). In the context of the shape from shading problem we are only concerned with first-order PDEs.

The following definitions and results can be found in Barles's, Bardi and Capuzzo Dolcetta's or Lions's books [2, 1, 25].

2.1.1 A notion of weak solutions

Definitions

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function (Ω is an open set of $\mathbb{R}^n$) and consider a Hamilton-Jacobi equation of the form:

$$H(x, u(x), \overrightarrow{\nabla u}(x)) = 0, \qquad x \in \Omega, \tag{2.1}$$

where H is a continuous scalar function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. H is called *Hamiltonian.*

Definition 1 *Viscosity subsolution:*
$u \in BUC(\Omega)$ (set of bounded and uniformly continuous functions) is a viscosity subsolution of equation (2.1) if:

$$\forall \phi \in C^1(\Omega)\ ,\ \forall x_0 \in \Omega \text{ local maximum of } (u - \phi),$$

$$H(x_0, u(x_0), \overrightarrow{\nabla}\phi\,(x_0)) \leq 0$$

Definition 2 *Viscosity supersolution:*
$u \in BUC(\Omega)$ is a viscosity supersolution of equation (2.1) if:

$$\forall \phi \in C^1(\Omega)\ ,\ \forall x_0 \in \Omega \text{ local minimum of } (u - \phi),$$

$$H(x_0, u(x_0), \overrightarrow{\nabla}\phi\,(x_0)) \geq 0$$

Definition 3 *Viscosity solution:*
u is a continuous viscosity solution of equation (2.1) if it is both a subsolution and a supersolution of (2.1).

Important practical **remarks**:

1. In practice, to check the hypotheses of these definitions, it can be useful to first fix the point x_0, and then to verify if, for all functions ϕ in $C^1(\Omega)$ such as x_0 is a local maximum (resp. minimum) of $(u - \phi)$, we have the appropriate inegality. Thus, in practice, to prove that u is a viscosity solution, we can only interest us to the particular points where the properties above are not trivial.

2. It is easy to prove that in the previous definitions, we can also impose that the test functions ϕ verify
$$\phi(x_0) = u(x_0).$$
Thus for example, the definition of subsolution can be rewritten as:

Definition 4 *$u \in BUC(\Omega)$ is a subsolution of (2.1) if :*
$\forall x_0 \in \Omega$

$$\forall \phi \in C^1(\Omega) \text{ such that}$$

$\phi(x_0) = u(x_0)$,
$\phi \geq u$ *on a neighbourhood of* x_0,

we have $H(x_0, u(x_0), \vec{\nabla}\phi\,(x_0)) \leq 0$.

Graphically this definition is much more intuitive than the previous one.

3. These definitions being based on local properties, even if it means reducing the size of the neighbourhood, *graphically* we can consider that the functions ϕ are affine.

Coherence of viscosity solutions

Viscosity solutions are weak solutions. They are not differentiable! Nevertheless, we are going to see that this notion is consistent with the notion of classical solutions.

Proposition 1 *Let u differentiable on Ω, a classical solution of (2.1). If $u \in BUC(\Omega)$, then u is a viscosity solution.*

△ Proof:
In fact, if u is differentiable, then for all functions $\phi \in C^1(\Omega)$ if x_0 is a local maximum (resp. minimum) of $(u - \phi)$, then

$$\overrightarrow{\nabla u}(x_0) = \vec{\nabla}\phi\,(x_0)$$

and then

$$H(x_0, u(x_0), \vec{\nabla}\phi\,(x_0)) = 0.$$

□
In other respects, we have also the following result:

Theorem 1 *Let u a viscosity solution of equation (2.1). if u is differentiable on Ω, then u is a classical solution.*

The reader can find a proof of this theorem in previous citations. □

In short, classical BUC solutions of (2.1) are differentiable viscosity solutions of (2.1). So viscosity solutions are generalized solutions of Hamilton-Jacobi equation.

Example: The eikonal equation

Throughout this chapter, we are going to illustrate our presentation with the particular case of the one-dimensional eikonal equation.

$$|\overrightarrow{\nabla u}(x)| - n(x) = 0, \quad \forall x \in]0,1[\tag{2.2}$$

where n is a real scalar function.

For the classical example where $n(x) = 1$ for all x in]0,1[,

$$|\overrightarrow{\nabla u}(x)| - 1 = 0, \quad \forall x \in]0,1[\tag{2.3}$$

it is easy to prove that the following function u is a viscosity solution:

$$u(x) = \begin{cases} x & \text{if } x \in]0, \frac{1}{2}[\\ 1 - x & \text{if } x \in [\frac{1}{2}, 1[. \end{cases} \tag{2.4}$$

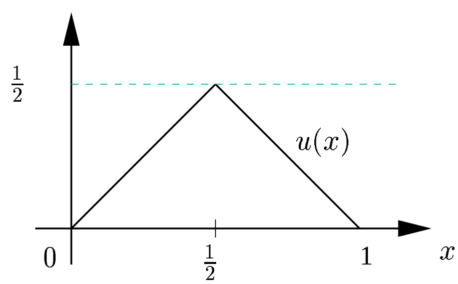
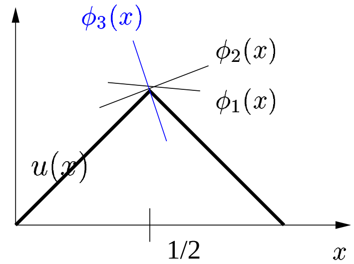


Figure 2.1: A solution of eikonal equation with n(x)=1.

△ Proof:

- For $x_0 \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$:
  On the subset $]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$ the function u gived by (2.4) is differentiable and is a solution of (2.2) in classical sense. Then by proposition 1, u is a viscosity solution on $]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$.

- For $x_0 = \frac{1}{2}$:
  - u is a subsolution:
    Suppose we have $\phi \in C^1(]0,1[)$ such as
    $u(\frac{1}{2}) = \phi(\frac{1}{2})$;
    and
    $$u \le \phi; \tag{2.5}$$
    in a neighbourhood of $\frac{1}{2}$ (see previous remarks).



ϕ_1 and ϕ_2 are two possible functions;
ϕ_3 can't verify $u(x) \leq \phi$ for every neighbourhood of $\frac{1}{2}$.

Figure 2.2: Illustration of: $\forall x \in V, u(x) \leq \phi(x) \Longrightarrow |\phi'(x)| \leq 1$

Graphically, identifying ϕ and its tangent at $\frac{1}{2}$, we can see clearly that this implies $|\phi'(\frac{1}{2})| \leq 1$ (see the same remarks and figure (2.2)).
Rigorously, for such a ϕ we have $(u-\phi)$ negative and in $C^1([0, \frac{1}{2}])$, so $\phi'(\frac{1}{2}) \leq 1$.
$\triangle$ In fact, if $(u-\phi)'(\frac{1}{2}) < 0$, since $(u-\phi) \in C^1([0, \frac{1}{2}])$, $(u-\phi)'(x) < 0$ on a neighbourhood W of $\frac{1}{2}$ in $[0, \frac{1}{2}]$; then $(u-\phi)$ decrease strictly. Since $(u-\phi)(\frac{1}{2}) = 0$ then $(u-\phi)(x) > 0$ on W. This contradicts $u \leq \phi$ imposed by (2.5). Thus $\phi'(\frac{1}{2}) \leq 1$.
$\square$
Similary, considering the subset $[\frac{1}{2}, 1]$, we can see that $\phi'(\frac{1}{2}) \geq -1$. So considering both the right and left derivatives, we conclude that $|\phi'(\frac{1}{2})| \leq 1$. We have obtained the desired inequality and u is a subsolution of (2.3).

- u is a supersolution:
  Graphically or using the same tools, it is easy to prove that there does not exist a function ϕ such $\phi(\frac{1}{2}) = u(\frac{1}{2})$ and $\phi \leq u$ on a neighbourhood of $\frac{1}{2}$. Then u is consequently a supersolution.

$\square$

Remarks:

1. In the case of the eikonal equation (2.3), it is very easy to prove that there can't exist solution with a downward kink; such a function can't be a supersolution (see figure (2.3)). This remark can be generalized to the eikonal equation (2.2).

2. We have said that viscosity solutions are generalized solutions of Hamilton-Jacobi equations, but all the generalized solutions are not viscosity solutions. In fact, there are many generalized solutions wich are not viscosity solutions: take for example the function

$$v(x) = \begin{cases} -x & \text{if } x \in]0, \frac{1}{2}[\\ -1+x & \text{if } x \in [\frac{1}{2}, 1[\end{cases} \tag{2.6}$$

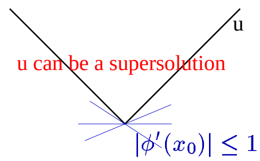


Figure 2.3: : There can't exist solutions with a downward kink

v could be a generalized solution of (2.3) in the same way as u defined by (2.4). But v is not a viscosity solution (see previous remark)!

3. It is important to note that there don't exist classical solutions of the equation (2.3). In general, the Hamilton-Jacobi equations are not well-posed in the classical sense: there don't exist solutions. As we will see, in the viscosity context, hypotheses of existence theorems are very weak. This is one of the most important advantages of the viscosity solutions.

2.1.2 Existence and uniqueness of continuous viscosity solutions

As we have seen in section (1.2), the "classical" modelisation of Shape-from-shading problem results in an Hamilton-Jacobi equation in the form:

$$H(x, \overrightarrow{\nabla u}(x)) = 0;$$

Thus we will limit ourselves to Hamiltonians which do not depend on u. In such a case, it is clear that to have uniqueness we must add boundary conditions. Our choice turns to Dirichlet conditions: in the following we will be interested in particular in the following problem:

$$\begin{cases} H(x, \overrightarrow{\nabla u}(x)) = 0 \text{ on } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases} \tag{2.7}$$

with Ω a bounded regular and convex open set of $\mathbb{R}^n$, and φ a scalar function defined on $\partial\Omega$.

Uniqueness result

We present here an uniqueness result due to Ishii [21]; This result has been proved later in a different manner by Lions [25]. Rouy and Tourin have also given this uniqueness result for Halmiltonians H which do not depend upon u (see [29]). For more general conditions, see [26].

Theorem 2 *let* $u, v \in BUC(\overline{\Omega})$ *respectively subsolution and supersolution of the equation:*

$$H(x, \overrightarrow{\nabla u}(x)) = 0 \textit{ on the bounded open } \Omega \subset \mathbb{R}^2 \ . \tag{2.8}$$

If the following hypotheses are verified:

1. $\forall x, y \in \Omega,\ \forall p \in \mathbb{R}^2,\ |H(x,p) - H(y,p)| \leq \omega(|x-y|(1+|p|))$, *where* ω *is a continuous nondecreasing function such that* $\omega(0) = 0$,
2. H *is continuous in* $\overline{\Omega} \times \mathbb{R}^2$ *and convex with respect to* $\overrightarrow{\nabla u}$,
3. *there exists a strict viscosity subsolution* $\underline{u} \in C^1(\Omega) \cap C(\overline{\Omega})$ *of (2.8) (i.e. such that* $H(x, \nabla \underline{u}(x)) < 0$ *for all* x *in* Ω*);*

then there exists at most one continuous viscosity solution of (2.8) verifying $u = \varphi$ *in* $\partial\Omega$, *where* $\varphi \in C(\partial\Omega)$ *(it is to say a continuous viscosity solution of (2.7)).*

Remark:

- Let a function $G : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m : x \longmapsto G(x)$; with n and m in $\mathbb{N}$. A sufficient condition to have existence of a nondecreasing continuous function ω such as $\omega(0) = 0$ and
$$\forall x, y \in E,\ |G(x) - G(y)| \leq \omega(|x-y|)$$
is *G uniformly continuous.*

  $\triangle$ Proof:
  Let $G : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m : x \longmapsto G(x)$ be an uniformly continuous function. Let note $\omega_G(\varepsilon) = sup\{|G(x) - G(y)|; |x-y| \leq \varepsilon\}$. Since G is uniformly continuous, the reader will verify that:

  - $\omega_G(\varepsilon) < \infty$,
  - $\omega_G(0) = 0$
  - ω_G is nondecreasing.

  Also, we have
$$|G(x) - G(y)| \leq \sup\{|G(z_1) - G(z_2)|; |z_1 - z_2| \leq |x-y|\} = \omega_G(|x-y|)$$
  $\square$

  In the particular case of $H : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ we have: if H uniformly continuous then there exists a nondecreasing continuous function ω such as $\omega(0) = 0$ and
$$\forall x, y \in \Omega,\ \forall p \in \mathbb{R}^n,\ |H(x,p) - H(y,p)| \leq \omega(|x-y|(1+|p|))$$

- Let (E,d) be a metric set. A sufficient condition to have $G : (E,d) \longrightarrow (E,d) : x \longmapsto G(x)$ uniformly continuous is *G continuous and (E,d) compact* (theorem of Heine).

- Let $(E, ||.||)$ be a normal vectorial space. A sufficient condition to have $G : (E, ||.||) \longrightarrow (E, ||.||) : x \longmapsto G(x)$ uniformly continuous is *G Lipschitz continuous.*

Recall:
In the eikonal equation case, it is easy to show that theorem 2 is sharp: there exists many solutions if there are singular points in Ω (see remark II.5.11 of [1]). Recall that in this context, a singular point is a point $x \in \Omega$ such as $n(x) = 0$.

Existence result for convex coecive Hamiltonians: value functions and boundary compatibility conditions

In this section, we are going to give not only an existence result but also an explicit expression of the solution as value function.
More exactly, we recall here a theorem of existence of viscosity solutions in the special case where the Hamiltonian H appearing in equation (2.7) (so with Dirichlet boundary conditions) is convex with respect to $\overrightarrow{\nabla u}$.
It is important to note that this existence theorem can be interpreted as giving compatibility constraints for the boundary conditions.

We note H^* the Legendre transform of H:

$$H^*(x, q) = \sup_{p \in \mathbb{R}^2} \{p.q - H(x, p)\} \ \leq +\infty.$$

The following theorem 3 is a special case of theorem 5.3 in [25].
Let us define $\forall x, y \in \overline{\Omega}$,

$$L(x, y) = \inf_{\xi \in C_{x,y}, T_0 > 0} \{ \int_0^{T_0} H^*(\xi(s), -\xi'(s)) ds \ \ \}$$

where $C_{x,y}$ is the set of ξ such as

- $\xi(0) = x$,
- $\xi(T_0) = y$,
- $\forall t \in [0, T_0], \xi(t) \in \overline{\Omega}$,
- $\xi' \in L^\infty(0, T_0)$.

Theorem 3 *If*

1. *$H \in C(\overline{\Omega} \times \mathbb{R}^2)$ is convex with respect to $\overrightarrow{\nabla u}$ for all x in $\overline{\Omega}$,*
2. *$H(x, p) \to +\infty$ when $|p| \to +\infty$ uniformly with respect to $x \in \overline{\Omega}$,*

3. *$inf_{p\in\mathbb{R}^2} H(x,p) \leq 0$ in $\overline{\Omega}$,*

4. $\forall x, y \in \partial\Omega\ ,\ \varphi(x) - \varphi(y) \leq L(x,y)$;

then the function u defined in $\overline{\Omega}$ by:

$$\begin{aligned} u(x) &= \inf_{y\in\partial\Omega}\{\varphi(y) + L(x,y)\} \\ &= \inf\{\int_0^{T_0} H^*(\xi(s), -\xi'(s))ds + \varphi(\xi(T_0))\} \end{aligned} \tag{2.9}$$

is a continuous viscosity solution of equation (2.7) (in particular u verifies $u(x) = \varphi(x)$ for all x in $\partial\Omega$).

This theorem implies the existence of a continuous viscosity solution of (2.7) and gives us an explicit solution.

Remark: Under hypotheses (1)-(2)-(3) of this theorem, we have then the following NSC (necessary and sufficient condition):
u defined by (2.9) is a viscosity solution of (2.7) iff

$$\forall x, y \in \partial\Omega\ ,\ \varphi(x) - \varphi(y) \leq L(x,y). \tag{2.10}$$

We will say that φ verifies the *compatibility condition* if (2.10) is verified.

2.2 Discontinuous viscosity solutions

2.2.1 Definitions

In this section we recall the notion of the discontinuous viscosity solutions; this notion of discontinuous solutions is due mostly to Ishii [23, 22] and is covered in detail in the book of Barles [2]. The recent book of Bardi and Capuzzo Dolcetta [1] synthesizes some recent results.

For the continuous viscosity solutions, we have kept a classical frame of mind. More exactly, we have considered an equation on a *open subset* and we have add a boundary condition. Now we are going to upset this traditional sketch:

Let us consider the following equation on the closed subset $\overline{\Omega}$:

$$F(x, u(x), \overrightarrow{\nabla u}(x)) = 0 \tag{2.11}$$

where F is defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ and only supposed locally bounded (F is not supposed continuous). The idea is to consider both the equation and the boundary condition.
Generally F is defined by:

$$F(x,u,p) = \begin{cases} H(x,u,p) & \text{for x in } \Omega, \\ G(x,u,p) & \text{for x in } \partial\Omega, \end{cases} \tag{2.12}$$

where H is a continuous function on $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2$ and G is a continuous function on $\partial\Omega\times\mathbb{R}\times\mathbb{R}^2$. For example, in the case of the Dirichlet condition, we can take:

$$F(x,u,p) = \begin{cases} H(x,u,p) & \text{for x in } \Omega, \\ u(x) - \varphi(x) & \text{for x in } \partial\Omega, \end{cases} \tag{2.13}$$

with φ continuous on $\partial\Omega$.

Before giving the definition of the discontinuous viscosity solutions, recall the definitions of the upper and lower semicontinuous envelopes. For better understand the following notions, the reader can refer to the chapter V of [1].

Definition 5 *Let u be a locally bounded function on a set E.*
$\forall x \in E$, let us note:

$$u^*(x) = \limsup_{y\to x} u(y)$$

$$u_*(x) = \liminf_{y\to x} u(y)$$

u^* et u_* are respectively call the upper semicontinuous envelope and lower semicontinuous envelope of u.

We recall also that $u : E \to \mathbb{R}$ is a upper (respectively, lower) semicontinuous (u.s.c, resp. l.s.c) if for any $x \in E$ and $\epsilon > 0$ there exist a δ such that for all $y \in E \cap B(x,\delta)$ $u(y) < u(x)+\epsilon$ (respectively, $u(y) > u(x) - \epsilon$). To familiarize oneself with these last notions, the reader can refer to the sections V-1 and V-2.1 of [1].

Definition 6 *A locally bounded function, u.s.c on $\overline{\Omega}$, v is a discontinuous viscosity subsolution of equation (2.11) if:*

$$\forall\phi \in C^1(\overline{\Omega})\ ,\ \forall x_0 \in \overline{\Omega} \text{ local maximum of } (v-\phi), \quad F_*(x_0, v(x_0), \overrightarrow{\nabla}\phi\,(x_0)) \le 0.$$

A locally bounded function, l.s.c on $\overline{\Omega}$, v is a discontinuous viscosity supersolution of equation (2.11) if:

$$\forall\phi \in C^1(\overline{\Omega})\ ,\ \forall x_0 \in \overline{\Omega} \text{ local minimum of } (v-\phi), \quad F^*(x_0, v(x_0), \overrightarrow{\nabla}\phi\,(x_0)) \ge 0.$$

Finally, a locally bounded function, u is a discontinuous viscosity solution of (2.11) if u^* is a subsolution and u_* is a supersolution of (2.11).

For the Dirichlet problem (2.13), H and φ being continuous, it is easy to caculate the functions F^* and F_*. We have:

$$\begin{array}{ll} F^*(x) = F_*(x) = H(x) & \text{if } x \in \Omega, \\ F^*(x) = max(H(x,u,\overrightarrow{\nabla u}), u(x) - \varphi(x)) & \text{if } x \in \partial\Omega, \\ F_*(x) = min(H(x,u,\overrightarrow{\nabla u}), u(x) - \varphi(x)) & \text{if } x \in \partial\Omega. \end{array}$$

For more details, we advise the reader to read chapter 4 of Barles's book [2].

2.2.2 Remarks and Examples

1. "In general discontinous viscosity solutions are continuous on Ω!"

   (a) More exactly, let us consider a Hamilton-Jacobi equation in one dimension with a (uniformly) coercive Hamiltonian, i.e. $H(x,p) \longrightarrow +\infty$ when $p \to +\infty$ (uniformly with respect to x).
   Let us give the following definition:

   Definition 7 *We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ is* piecewise C^1 *if there exists a subdivision $a = a_0 < a_1 < a_2 < \ldots < a_n = b$ of [a,b] such that the restriction $u_{|]a_i,a_{i+1}[}$ of u to $]a_i,a_{i+1}[$ is $C^1(]a_i,a_{i+1}[)$ and can be continued to a $C^1([a_i,a_{i+1}])$ function.*

   Let us emphasize that with this definition, the piecewise C^1 functions can be discontinuous. (see figure (2.4)).

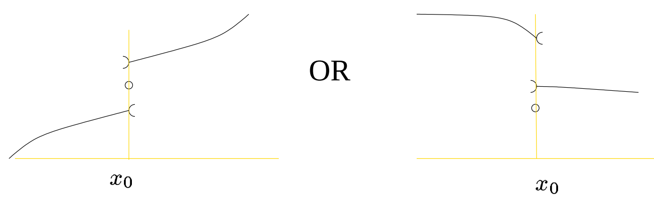


Figure 2.4: Examples of discontinuous piecewise C^1 functions.

   We prove that with a coercive Hamiltonian, the piecewise C^1 viscosity solutions (in the discontinuous sense) can't be discontinuous in $\Omega =]a,b[$.
   $\triangle$ Proof: Let x_0 be in Ω; by coercivity of H, there exists a real M such that for all p with $|p| > M$ we have $H(x_0,p) > 0$. Let u be a piecewise C^1 function having a discontinuity at x_0. If u is a solution, u^* must be a subsolution. Let us consider the two cases illustrated in figure (2.5). Since $u'(.)$ is bounded, it is graphically clear that there exists a function $\phi \in C^1(\Omega)$ such that:

   - $\phi \geq u^*$ in a neighbourhood of x_0,
   - $\phi(x_0) = u^*(x_0)$,
   - $|\phi'(x_0)| > M$.

   In particular, $H(x_0,\phi'(x_0)) > 0$ and u^* can't be a subsolution.
   In the same way, we can deal with the other cases (which are not represented in the figure (2.5)) when $\lim_{x\to x_0^+} u(x) < \lim_{x\to x_0^-} u(x)$. $\square$

   (b) We can be even more precise:
   In one dimension, for a piecewise C^1 solution to be discontinuous at x_0, it is necessary that H verifies:

$$\exists A \in \mathbb{R} \quad | \quad \forall q \in [A, +\infty[\quad H(x_0, q) = 0$$

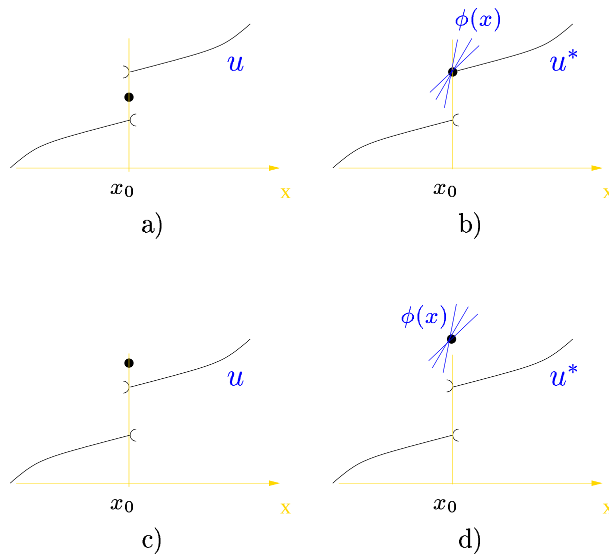
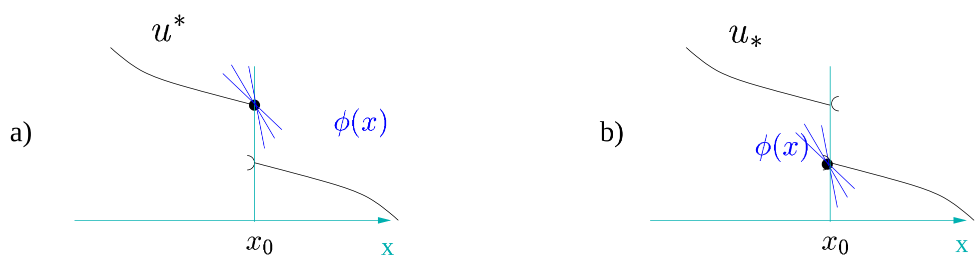


Figure 2.5: u^* can't be a subsolution

or

$$\exists A \in \mathbb{R} \quad | \quad \forall q \in]-\infty, A] \quad H(x_0, q) = 0.$$

The reader will convince him(her)self of this by studying figure (2.6).



$$\begin{array}{lll} \text{a)} \quad u^* \text{ subsolution} & \Longrightarrow & \forall q \in]-\infty, A], H(x_0, q) \leq 0, \\ \text{b)} \quad u_* \text{ supersolution} & \Longrightarrow & \forall q \in]-\infty, A], H(x_0, q) \geq 0. \end{array}$$

Figure 2.6: For a piecewise C^1 solution to be discontinuous at x_0, H must be null on $\{x_0\} \times]-\infty, A]$ or $\{x_0\} \times [A, +\infty[$

(c) Nevertheless discontinuities can appear when the limit

$$\lim_{x \to x_0^+} u'(x)$$

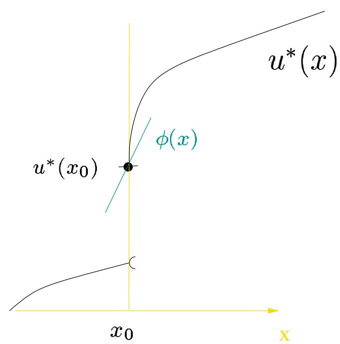
or

$$\lim_{x \to x_0^-} u'(x)$$

does not exist in $\mathbb{R}$. This is the case, for example, when u is $C^1([a, x_0[\cup]x_0, b])$ with

$$\lim_{x \to x_0^+} u'(x) = +\infty$$

(this is impossible when H is uniformly coercive); remark that in this case, it is clear that u^* is a subsolution because there doesn't exist $\phi \in C^1([a, b])$ such that $\phi(x_0) = u^*(x_0)$ and $\phi \geq u^*$ on a neighbourhood of x_0 (see figure 2.7).



If a solution u verifies $\lim_{x \to x_0^+} u'(x) = +\infty$,
it can present a discontinuity at x_0.

Figure 2.7: discontinuities are possible when $\lim_{x \to x_0^+} u'(x) = +\infty$.

Remarks:

i. In theory in the eikonal case, this kind of problem mustn't appear. In effect, the definition of "viscosity solutions" needs to impose F locally bounded (see section V.2.1 of [1]); in fact, here we have H continuous. Thus the function n is locally bounded. Nevertheless, in "SFS" problem the brightness I(x) can be null ($\Rightarrow n(x) = +\infty$) and so we would like to be able to deal with no locally bounded Hamiltonians. In the article [26], P.L.Lions, E.Rouy and A. Tourin attempt to answer to this theorical question.

ii. We can generalised these results to discontinous Hamiltonian.

iii. This kind of results can also be generalised to Hamiltonians defined on a two dimensional set Ω.

2. Types of Discontinuities on the boundary for the eikonal equation.
   As we have shown above, in general, discontinuous viscosity solutions are continuous in Ω. Nevertheless, solutions can be discontinuous on the boundary, but all types of discontinuities cannot occur.
   To illustrate this, let us classify the possible discontinuities for the one-dimensional eikonal equation (on the interval $\overline{\Omega} = [0,1]$). In particular let us look at the possible discontinuities at the point $x_0 = 0$. To simplify, let us consider the solution u such that:

   - $u \in C^1(]0,1[)$,
   - $u(x) \longrightarrow l$ when $x \to 0^+$,
   - $u(0) = \varphi(0)$

   Reminder: To prove that u is a solution (in the discontinuous sense), we must prove:

   (a) u^* is a subsolution:
   i.e: we have
   - $u^*(x_0) \leq \varphi(x_0)$

   OR
   - $\forall \phi \in C^1([0,1])$ such that
     - $\phi(x_0) = u^*(x_0)$,
     - and $\phi \geq u^*$ on a neighbourhood of x_0;

     we have $|\phi'(x_0)| \leq n(x_0)$.

   (b) u_* is a supersolution:
   i.e.: we have
   - $u_*(x_0) \geq \varphi(x_0)$

   OR
   - $\forall \phi \in C^1([0,1])$ such that
     - $\phi(x_0) = u_*(x_0)$,
     - and $\phi \leq u_*$ on a neighbourhood of x_0;

     we have $|\phi'(x_0)| \geq n(x_0)$.

   Let us also remember that n is supposed continuous on [0,1] and then $n(x_0)$ is the half-tangent of u at x_0.

   (a) A solution u can't present a discontinuity of kind represented in the figure (2.8-a).
   $\triangle$ proof:
   Graphically we see that $u_*(0) < \varphi(0)$ and that there exist $\phi \in C^1([0,1])$ such that

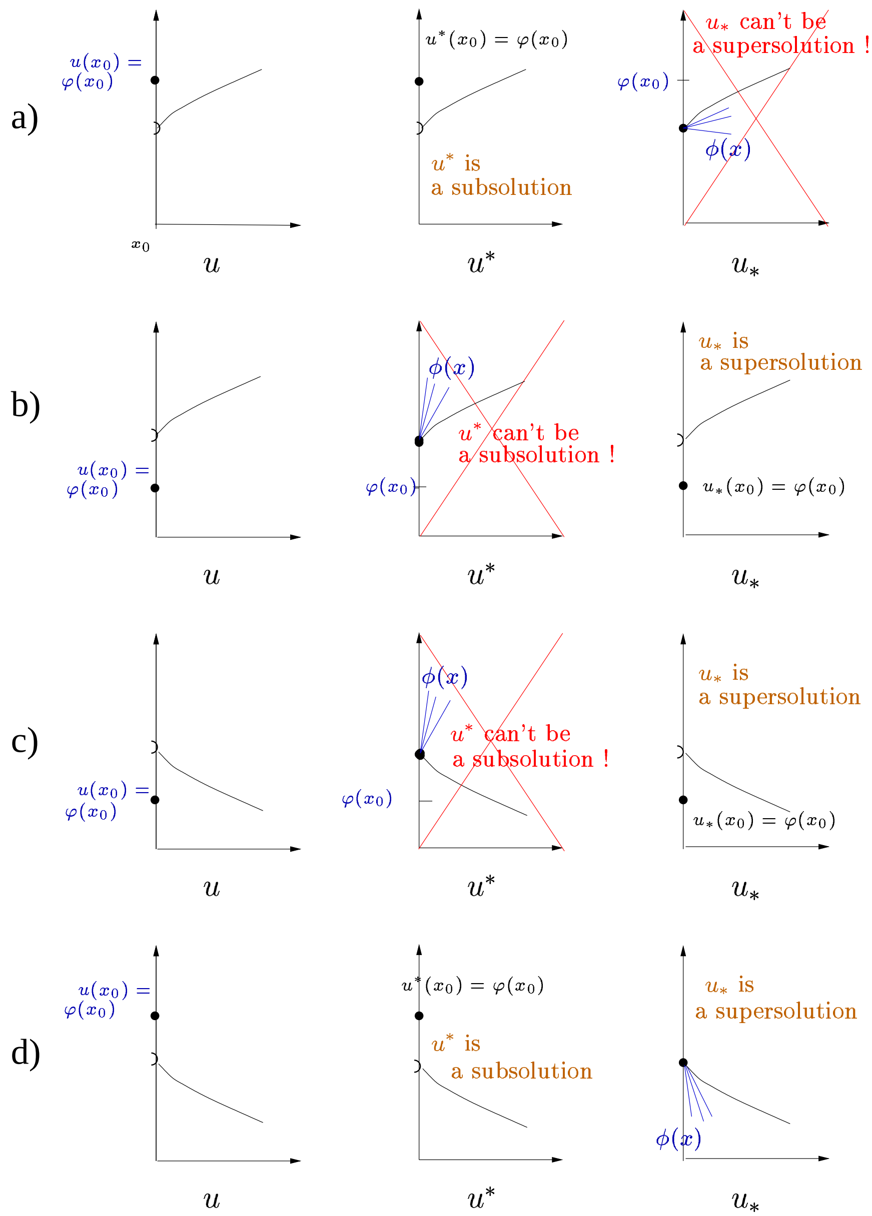


Figure 2.8: Possible discontinuities on the boundary of solutions for the eikonal equation $|u'(x)| - n(x) = 0$ in one dimension

- $\phi(0) = u_*(0)$,
- $\phi \leq u$,
- $|\phi'(0)| < n(0)$.

$\square$

(b) A solution u can't present a discontinuity of kind represented in the figure (2.8-b).
$\triangle$ proof:
Similary, graphically we see that $u^*(0) > \varphi(0)$ and that there exist $\phi \in C^1([0,1])$ such that

- $\phi(0) = u^*(0)$,
- $\phi \geq u$,
- $|\phi'(0)| > n(0)$.

$\square$

(c) A solution u can't present a discontinuity of kind represented in the figure (2.8-c).

(d) Only the discontinuities of kind represented in the figure (2.8-d) can be present.
$\triangle$ proof: We leave it to the reader.

The same results hold for $x_0 = 1$.

3. No there are compatibility conditions !
In the eikonal equation example, it is easy to see that the difference between the continuous and discontinuous solutions appears only on the boundary of the domain of definition. Unlike the continuous case where we need compatibility conditions, in the discontinous case, there exists solutions for all boundary conditions (see the subsection (2.2.4)). In figure (2.9) we show discontinuous solutions of the viscosity eikonal equation for different boundary conditions.

4. Solutions with discontinuous Hamiltonian with respect to the space variable x:
The definition 6 above applies to discontinuous Hamiltonians; but it is important to remark that nearly all theorems of the viscosity solutions theory need hypotheses of regularity on the Hamiltonian H. Some examples follow.

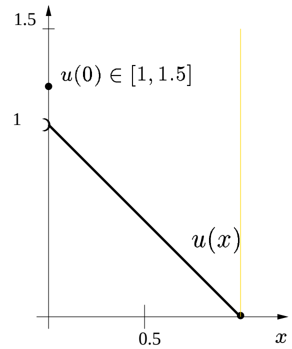
2.2.3 A uniqueness result

Let us recall the following usual definition:

Definition 8 *We say that we have a* maximum principle *for the Hamilton-Jacobi equation*

$$H(x, u(x), \overrightarrow{\nabla u}(x)) = 0 \textit{ in an } \underline{\textit{open}} \textit{ set } \Omega, \tag{2.14}$$

when we have:
"for all subsolution u and supersolution v defined on $\overline{\Omega}$, $u \leq v$ on $\partial\Omega \Longrightarrow u \leq v$ on $\overline{\Omega}$".

<table>
<tr><th></th><th>Boundary conditions</th><th>solution</th></tr>
<tr><td>a)</td><td>$\varphi(0) = 1.5,$
$\varphi(1) = 0;$</td><td>
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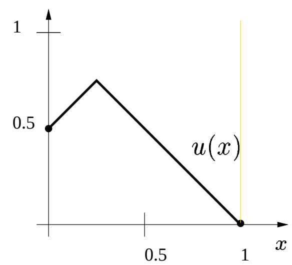
<tr><td>b)</td><td>$\varphi(0) = 0.5,$
$\varphi(1) = 0;$</td><td>
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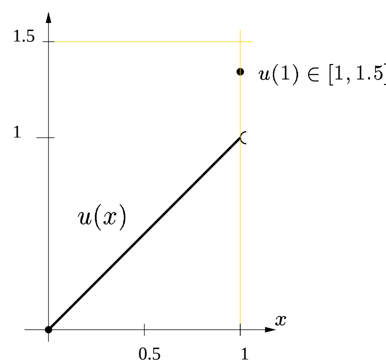
<tr><td>c)</td><td>$\varphi(0) = 0,$
$\varphi(1) = 1.5;$</td><td>
</td></tr>
</table>

Figure 2.9: Examples of discontinuous viscosity solutions of the eikonal equation $|u'(x)| - 1 = 0$ with different boundary conditions.

Remarks: In the continuous case, the maximum principle involves the uniqueness of the solution of the Dirichlet problem.

△ Proof: Let u,v be two continuous viscosity solutions to the Dirichlet problem. Since for all x in $\partial\Omega$, u(x)=v(x) (Dirichlet condition) and u,v are both subsolution and supersolution, the maximum principle implies that u≤v and v≤ u on $\overline{\Omega}$, hence the conclusion.

□

Under the hypotheses "HNCL" (described below) it is known that we have a maximum principle for discontinuous viscosity solutions (see theorem 4.2 of Guy Barles's book [2]). For example, this result applies to the eikonal equation (2.3) with Dirichlet conditions; nevertheless we don't have the uniqueness of the solution! In fact, for all $y \in [1, 2]$, the function u_y defined by (see figure 2.10)

$$u_y(x) = \begin{cases} x & \text{if } x \in [0, 1[, \\ y & \text{if } x = 1; \end{cases}$$

is a solution of the equation (2.3) with the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 2$. So,

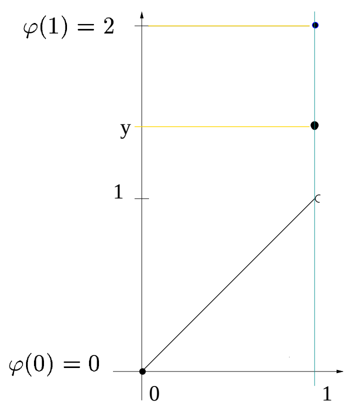


Figure 2.10: For all $y \in [1, 2]$, u_y is a solution of (2.3).

in the discontinuous case, to have uniqueness we need a stronger property:

Definition 9 *We say that we have the strong uniqueness property for the equation*

$$F(x, u(x), \overrightarrow{\nabla u}(x)) = 0 \text{ in a } \underline{\text{set}}\ E, \tag{2.15}$$

when we have:
"for all subsolution u and supersolution v, $u \leq v$ in E".

We have the following strong uniqueness result (see [2]):

Theorem 4 *if H satisfies the hypotheses "HNCL" (defined below) and if there exists a constant $C > 0$ such that for all p in $\mathbb{R}^2$,*

- $\forall x \in \partial\Omega, \quad H(x, p + \lambda\eta(x)) \leq 0 \quad \Longrightarrow \quad \lambda \leq C(1 + |p|),$
  where $\eta(x)$ is the unit outward pointing normal vector to $\partial\Omega$.
- $H(x, p - \lambda\eta(x)) \to +\infty$ *uniformly with respect to x, when $\lambda \to +\infty$;*

then for all subsolution u and supersolution v (in the discontinuous sense), $u \leq v$ in Ω.

The hypotheses "HNCL" are:

1. There exists a function m_R which goes to zero at zero, such that $\forall x, y \in \Omega, \ \forall p \in \mathbb{R}^2, \ |H(x,p) - H(y,p)| \leq m_R(|x - y|(1 + |p|))$
2. For all x in Ω, $H(x,p)$ is convex with respect to p.
3. There exists a function Φ of class C^1 in Ω, continuous in $\overline{\Omega}$ such that $H(x, D\Phi(x)) \leq \delta < 0$ in Ω
4. There exists a function m_R which goes to zero at zero, such that $\forall x \in \partial\Omega$, $\forall p, q \in \mathbb{R}^2$, $|H(x,p) - H(x,q)| \leq m_R(|p - q|)$.

Remarks:

1. For the Dirichlet problem, if the above hypotheses are verified, we then have uniqueness of discontinuous solutions in Ω, but not in $\overline{\Omega}$! That means that if u and v are two solutions, then for all x in Ω, $u(x) = v(x)$; but for all x in $\partial\Omega$, $u(x)$ can be different from $v(x)$.
2. We have stated the theorem in the case where the Hamiltonian H doesn't depend of u; for the more general case, see [2].
3. We can note that the hypotheses "HNCL" are very close to the hypotheses of the uniqueness theorem in the continuous case (theorem 2).

Proposition 2 *Let F be a function such that the strong uniqueness property is verified for the associated equation (see definition 9) and let u a solution of this equation.*
Then we can claim that u is continuous on the adequate set.

$\triangle$ Proof: By definition of u^* and u_* we have $u_* \leq u^*$. The strong uniqueness property involves $u^* \leq u_*$. Then $u^* = u_*$. u is continuous. $\square$

Proposition 3 • *A sufficient condition for the hypothesis*

$$\forall x \in \partial\Omega, \quad H(x, p + \lambda\eta(x)) \leq 0 \quad \Longrightarrow \quad \lambda \leq C(1 + |p|),$$

to be satisfied is:

$$H(x,q) \leq 0 \Longrightarrow |q| < C.$$

- *A sufficient condition for the hypothesis*

$$H(x, p - \lambda\eta(x)) \to +\infty \quad \textit{uniformly with respect to x, when } \lambda \to +\infty,$$

to be satisfied is "$H(x, q)$ is coerciv in q uniformly with respect to x", i.e:

$$H(x, q) \to +\infty \quad \textit{uniformly with respect to x, when } q \to +\infty$$

$\triangle$ Proof: If $H(x, q) \leq 0 \Longrightarrow |q| < C$ (we can take $C > 1$) then

$$\forall x \in \partial\Omega, \quad H(x, p + \lambda\eta(x)) \leq 0 \quad \Longrightarrow (p + \lambda\eta(x)) \leq C$$

then $|\lambda||\eta(x)| \leq C + |p|$. since $|\eta(x)| = 1$ we have $\lambda \leq C(|p| + 1)$. $\square$

2.2.4 An existence result

There exists a lot of existence theorems, but here, we only give the theorem we need. The reader can find the following theorem in Bardi and Capuzzo Dolcetta's book [1] (theorem V.4.13), it deals with Hamilton-Jacobi-Bellman equations.

Before giving the theorem, let us state our assumptions we need:

(A_0) $\left\{ \begin{array}{c} A \text{ is a topological space,} \\ f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{is continuous;} \end{array} \right.$

(A_1) f is bounded on $B(0, R) \times A$ for all $R > 0$;

(A_2) there exists a modulus ω_f such that

$$|f(y, a) - f(x, a)| \leq \omega_f(|x - y|, R),$$

for all $x, y \in B(0, R)$ and $R > 0$,
(a modulus is a function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $R > 0$, $\omega(., R)$ is continuous, nondecreasing, and $\omega(0, R) = 0$)

(A_3) f is Lipschitz continuous in x uniformly in a,

(A_4) Let $l : \mathbb{R}^N \times A \to \mathbb{R}$ such that:

$$\left\{ \begin{array}{l} l \text{ is continuous;} \\ \text{there exists a modulus } \omega_l \text{ and a constant } M \text{ such that} \\ |l(x, a) - l(y, a)| \leq \omega_l(|x - y|) \text{ and } |l(x, a)| \leq M, \quad \forall x, y \in \mathbb{R}^2 \text{ and } a \in A; \end{array} \right.$$

Theorem 5 *Let $H(x, p) = \sup_{a \in A}\{-f(x, a) \cdot p - l(x, a)\}$; assume the hypotheses (A_0)-(A_4) are satisfied and let $\varphi \in BC(\partial\Omega)$, then u defined by*

$$u(x) = \inf_{\xi : \mathbb{R}+ \to A} \int_0^{t_x(\xi)} l(y_x(s), \xi(s))e^{-\lambda s} ds + e^{-\lambda t_x(\xi)}\varphi(y_x(t_x(\xi))),$$

(where y_x is the solution of the differential equation

$$\begin{cases} y'(t) = f(y(t), \xi(t)), & t > 0, \\ y(0) = x, \end{cases}$$

and $t_x(\xi)$ is the first time the trajectory $y_x(., \xi)$ goes out of $\overline{\Omega}$)
is a viscosity solution in the <u>discontinuous sense</u> of

$$\begin{cases} \lambda u + H(x, \overrightarrow{\nabla u}) = 0 & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega \end{cases} . \tag{2.16}$$

Let us emphasize the fact that the theorem is true even if $\lambda = 0$. In this case ($\lambda = 0$) and if A is compact, we have the corollary:

Corollary 1 *If $H(x,p) = \sup_{a \in A}\{-f(x,a) \cdot p - l(x,a)\}$, with $A \subset \mathbb{R}^2$ compact, $f : \mathbb{R}^2 \times A \to \mathbb{R}^2$ Lipschitz continuous in x uniformly in a, with $l : \mathbb{R}^2 \times A \to \mathbb{R}$ continuous such that there exists constants ω_l and M such that*
$|l(x,a) - l(y,a)| \leq \omega_l(|x-y|)$ and $|l(x,a)| \leq M, \quad \forall x, y \in \mathbb{R}^2$ and $a \in A$;
then u defined by

$$u(x) = \inf_{\xi \in \mathcal{A}} \int_0^{t_x(\xi)} l(y_x(s), \xi(s)) ds + \varphi(y_x(t_x(\xi))), \tag{2.17}$$

(where y_x and $t_x(\xi)$ are defined as above)
is a viscosity solution of

$$\begin{cases} H(x, \overrightarrow{\nabla u}) = 0 & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega \end{cases} . \tag{2.18}$$

with Dirichlet conditions in the discontinuous viscosity solutions sense.

In general in practice, the Hamiltonian H is not given in the form

$$H(x,a) = \sup_{a \in A}\{-f(x,a) \cdot p - l(x,a)\}$$

but as an explicit form; for example $H(x,p) = I(x)\sqrt{1+|p|^2} + p \cdot \mathbf{l} - \gamma$ (see chapter 4).
In the case where $H(x,p)$ is a convex continuous function in p (and Lipschitz continuous at least locally in x) then it is possible to write $H(x,p)$ as a supremum of affine functions: We note H^* the dual convex function of H (also called the "Legendre transform" of H):

$$H^*(x,a) = \sup_{p \in \mathbb{R}^2}\{p.a - H(x,p)\} \ \leq +\infty.$$

We know that

$$H(x,p) = \sup_{a \in Dom(H^*(x,.))}\{p.a - H^*(x,a)\}$$

(see for example [12]).
Nevertheless, we can remark that in general $D_x := Dom(H^*(x,.))$ depends on x. To remove this dependency we need to change variables.
Let us take again the example $H(x,p) = I(x)\sqrt{1+|p|^2} + p\cdot \mathbf{l} - \gamma$; Through differential calculus, we can compute H^* explicitly :

$$H^*(x,a) = \begin{cases} -\sqrt{I(x)^2 - |a-\mathbf{l}|^2} + \gamma & \text{if } x \in \overline{B}(\mathbf{l}, I(x)) \\ +\infty & \text{otherwise.} \end{cases}$$

In this example, $D_x = \overline{B}(\mathbf{l}, I(x))$ depend of x. Therefore we have

$$H(x,p) = \sup_{a\in D_x}\{p.a + \sqrt{I(x)^2 - |a-\mathbf{l}|^2} - \gamma\},$$

and by the change of variables $b = (a-\mathbf{l})/I(x)$, we also have

$$H(x,p) = \sup_{b\in\overline{B}(0,1)}\{p.\underbrace{(I(x)b+\mathbf{l})}_{-f(x,b)} - \underbrace{(\gamma - I(x)\sqrt{1-|b|^2})}_{l(x,b)}\},$$

and H has the desired form.

In all cases where there exists a suitable change of variables, the reader can check that the function u given in the section (2.1) by

$$u(x) = \inf_{\xi}\int_0^{t_x(y_x)} H^*(y_x(s), -y_x'(s))ds + \varphi(y_x(t_x(y_x))),$$

where

$$\begin{cases} y_x(0) = x \\ y_x(t_x(y_x)) \in \partial\Omega. \\ \forall s \in [0, t_x(y_x)], y_x \in \overline{\Omega}, y_x'(s) \in L^\infty(0, t_x(y_x)). \end{cases}$$

is equal to the function u given above (2.17).

As in the continuous case with the theorem 3, one very nice thing about these theorems is that not only do they guarantee the existence of viscosity solutions, but they also give a solution expressed as a value function of an optimal control problem and provide a way of constructing one explicitely.

2.2.5 A stability result

The previous sections might have already convinced the reader of the theorical and practical interest of viscosity solutions; they have even more advantages: In point of fact, the notion of viscosity solutions also distinguishes by its important stability.

Theorem and corollary:

Let $(v_\epsilon)_{\epsilon>0}$ be a sequence of uniformly locally bounded functions defined on $\overline{\Omega}$. Let us define the two functions $\overline{v}$ and $\underline{v}$:

Definition 10 *$\forall x \in \overline{\Omega}$,*

$$\overline{v}(x) = \limsup_{\substack{\epsilon \to 0,\\ y \to x}} v_\epsilon(y)$$

$$\underline{v}(x) = \liminf_{\substack{\epsilon \to 0,\\ y \to x}} v_\epsilon(y)$$

We have the theorem (see Barles book [2]):

Theorem 6 (Stability of viscosity solutions) *Let F_ϵ be a sequence of uniformly locally bounded functions on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ (Ω being a open set of $\mathbb{R}^n$). Let us suppose that for all $\epsilon > 0$, u_ϵ is a subsolution (respectively a supersolution) of F_ϵ on $\overline{\Omega}$ and that the functions u_ϵ are uniformly locally bounded on $\overline{\Omega}$.*
Then

$$\overline{v}(x) = \limsup_{\substack{\epsilon \to 0,\\ y \to x}} v_\epsilon(y) \qquad \textit{(respectively } \underline{v}(x) = \liminf_{\substack{\epsilon \to 0,\\ y \to x}} v_\epsilon(y)\textit{)}$$

is a subsolution (a supersolution, respectively) of the equation

$$\underline{F}(x, u(x), \overrightarrow{\nabla u}(x)) = 0 \ \textit{on}\ \overline{\Omega} \qquad \textit{(respectively} \quad \overline{F}(x, u(x), \overrightarrow{\nabla u}(x)) = 0 \ \textit{on}\ \overline{\Omega}\textit{)};$$

where $\underline{F}(X) = \liminf_{\substack{\epsilon \to 0,\\ Y \to X}} F_\epsilon(Y)$ *($X, Y \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$)*
($\overline{F}(X) = \limsup_{\substack{\epsilon \to 0,\\ Y \to X}} F_\epsilon(Y)$, respectively).

In his book [2], G.Barles explains the interest of this theorem: it allows to pass to the limit and that almost without any hypotheses. The difficulty is then to relate the two objects $\overline{v}$ and $\underline{v}$. In some particular cases, this relation is obvious. For example, when the strong uniqueness property is true the previous result yields:

Corollary 2 *Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a locally bounded function verifying the strong uniqueness property on Ω (see definition 9). Let $(F_\epsilon)_{\epsilon>0}$ be a sequence of functions such that*

$$F_\epsilon \underset{\epsilon \to 0}{\longrightarrow} F$$

locally uniformly with respect to the other variables. Let u_ϵ be uniformly locally bounded functions such that for all $\epsilon > 0$, u_ϵ is a solution *(in the discontinuous sense) of*

$$F_\epsilon(x, u, \overrightarrow{\nabla u}) = 0 \ \textit{on}\ \overline{\Omega}.$$

Then, when ϵ vanishes to zero, the sequence u_ϵ converges on Ω toward a function u which is equal to the discontinuous viscosity solution of $F(x, u, \overrightarrow{\nabla u}) = 0$ on Ω.

$\triangle$ Proof of the corollary 2:

1. F being locally bounded and $(F_\epsilon)_{\epsilon>0}$ converging toward F locally uniformly, the functions F_ϵ are locally uniformly bounded.

2. We need the lemma:

   Lemma 1 *Let $(F_\epsilon)_{\epsilon>0}$ be a sequence of functions which converges* locally uniformly *toward a function F (i.e. for all X there exists a neighbourhood V_X such that the sequence (F_ϵ) converges uniformly toward F on V_X); then $\underline{F} = F_*$ and $\overline{F} = F^*$.*

   Proof of the lemma:
   $\triangle$ By definition of $F_*(X) := \liminf_{Y\to X} F(Y)$, for all $\delta > 0$, there exists a neighbourghood V_X of X such that for all $Y \in V_X$

   $$F_*(X) - \delta < F(Y).$$

   Even if we reduce V_X, since F_ϵ converges uniformly toward F on V_X, there exists a neighbourghood W_0 of 0 such that $\forall \epsilon \in W_0$ and for all $Y \in V_X$

   $$F(Y) - \delta < F_\epsilon(Y)$$

   and thus $\forall \epsilon \in W_0$ and for all $Y \in V_X$ we have:

   $$F_*(X) - 2\delta \;<\; F_\epsilon(Y).$$

   Taking the "liminf" we obtain $F_*(X) \le \underline{F}(X) + 2\delta$. This is true for all $\delta > 0$, therefore $F_*(X) \le \underline{F}(X)$ and the equality follows (because clearly $\underline{F}(X) \le F_*(X)$).

   Using the same idea, we also obtain $\overline{F} = F^*$. $\square$

   Remark:
   Without the hypothesis of "locally uniform convergence", this lemma is false even if the functions F and F_ϵ are continuous. For example, let us consider the function $F_\epsilon : [0,1] \to \mathbb{R} : x \mapsto F_\epsilon(x)$ whose graph is represented in figure 2.11. In this example we have

   - $F_\epsilon \longrightarrow 0$ when $\epsilon \to 0$,
   - F_ϵ and the zero function are continuous on $[0,1]$,

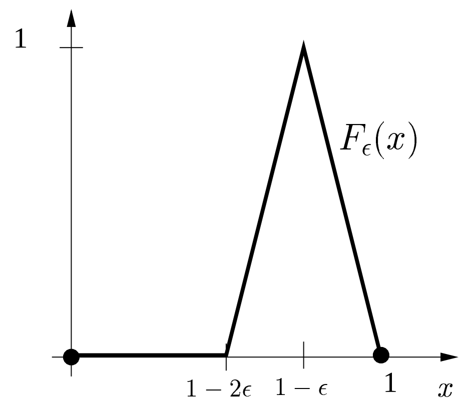


Figure 2.11: Counter-example for $\overline{F} = F^*$: the sequence $(F_\epsilon)_{\epsilon>0}$ does not converge uniformly toward 0 in the neighbourhoods of 1.

- nevertheless $\overline{F}(1) = 1$ and $F^*(1) = 0$.

3. Since $F_\epsilon \longrightarrow F$ when $\epsilon \to 0$ locally uniformly with respect to the other variables, the previous lemma implies $\underline{F} = F_*$ and $\overline{F} = F^*$. Thus, by applying theorem 6 to $v_\epsilon = u_\epsilon^*$ and $v_\epsilon = u_{\epsilon *}$, the functions
$$\overline{u} := \limsup_{\substack{\epsilon \to 0,\\ Y \to X}} \; {u_\epsilon}^*(Y)$$
and
$$\underline{u} := \liminf_{\substack{\epsilon \to 0,\\ Y \to X}} \; u_{\epsilon *}(Y)$$
are subsolution of $\underline{F} = F_*$ and supersolution of $\overline{F} = F^*$, respectively. The strong uniqueness property involves $\overline{u} \le \underline{u}$ on Ω. Therefore $\overline{u} = \underline{u}$ on Ω (we always have $\overline{u} \ge \underline{u}$). Let us note $u := \overline{u} = \underline{u}$ (a continuous function on Ω). We conclude:
   - u_ϵ converges toward u (when $\epsilon \to 0$);
   - u is equal to the viscosity solution of F in Ω.

□

Remark:

With the hypotheses of corollary 2, not only does the sequence $(u_\epsilon)_{\epsilon>0}$ converge toward u, but also the convergence is uniform on all compact set included in Ω. In effect, we have the proposition 4 (see lemma 4.1 of [2]):

Proposition 4 *If the functions* $\overline{u}$ *and* $\underline{u}$ *defined in theorem 6 are equal on* $\overline{\Omega}$ *(resp. in* Ω*) then the sequence* $(u_\epsilon)_\epsilon$ *converges uniformly on all compact of* $\overline{\Omega}$ *(resp. on all compact of* Ω*) toward the function* $u = \overline{u} = \underline{u}$.

Application:

In section 4.1.3 which deals with the SFS problem, we use corollary 2 to prove the robustness to noise of the results.

Chapter 3

Approximation schemes and numerical algorithms for solving Hamilton-Jacobi-Bellman equations (with null interest rate); convergence towards viscosity solutions.

In this chapter we develop tools for obtaining numerical approximations of viscosity solutions of Hamilton-Jacobi-Bellman (HJB) equations with null interest rate. The proposed method leads to finite difference schemes. We also explain how to design numerical algorithms from these approximation schemes. Finally we show the correctness of our algorithms and schemes by proving that they converge towards viscosity solutions of the HJB equation.

3.1 A method to get approximation schemes of a HJB equations with null interest rate

The Hamilton-Jacobi-Bellman equations with null interest rate are PDEs defined as follows:

$$\sup_{a \in A}\{-f(x,a) \cdot \overrightarrow{\nabla u} x - l(x,a)\} = 0 \qquad \text{if } x \in \Omega.$$

We add Dirichlet boundary conditions to this equation:

$$u(x) - \varphi(x) = 0 \qquad \text{if } x \in \partial\Omega.$$

The purpose of this section is to present a method for building finite difference approximation schemes for HJB equations with null interest rate. This method is based on the dynamic programming principle and yields approximation schemes of the form $g(x, a, b, c, d) = 0$.

Let Ω be a bounded open set, for example the rectangular domain $]0, X[\times]0, Y[$ of $\mathbb{R}^2$.

First, we remind the reader of the definition of a approximation scheme that we use in the sequel [3].

An approximation scheme is a functional equation (but not a PDE) of the form

$$S(\rho, x, u(x), u) = 0 \text{ in } \overline{\Omega};$$

where S is a function:

$$\begin{array}{lcl} \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \times B(\overline{\Omega}) & \longrightarrow & \mathbb{R} \\ (\rho, x, t, u) & \longmapsto & S(\rho, x, t, u). \end{array}$$

For example, suppose we want to approximate a function u such that its directional derivative in the direction $\overrightarrow{v}$ at point x is equal to $\lambda(x)$. We can use the following approximation scheme $S(\rho, x, u(x), u) = 0$ in $\overline{\Omega}$:

$$S(\rho, x, t, u) = \frac{u(x + \rho\overrightarrow{v}) - t}{\rho} - \lambda(x).$$

Let H be a convex Hamiltonian which can be written as a supremum

$$H(x, p) = \sup_{a \in A}\{-f(x, a) \cdot p - l(x, a)\}$$

and which verifies the hypotheses of theorem 5. Let u be defined by

$$u(x) = \inf_{y_x,\ t_x} \int_0^{t_x} H^*(y_x(s), -y_x'(s))ds + \varphi(y_x(t_x)),$$

where the inf is taken with respect to the set of $(y_x,\ t_x)$ such that

$$\begin{cases} y_x(0) = x \\ y_x(t_x) \in \partial\Omega. \\ \forall s \in [0, t_x], y_x(s) \in \overline{\Omega}, y_x' \in L^\infty(0, t_x). \end{cases}$$

By this theorem, u is a solution of the HJB equation with Dirichlet conditions:

$$\begin{cases} H(x, \overrightarrow{\nabla u}) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Since u is a value function, the dynamic programming principle allows us to write that for all $\tau \geq 0$, $u(x)$ is equal to the *inf* of

$$\int_0^{t_x \wedge \tau} H^*(y_x(s), -y_x'(s))ds + \varphi(y_x(t_x))\ \chi_{\{t_x \leq \tau\}}(t_x) + u(y_x(\tau))\ \chi_{\{t_x > \tau\}}(t_x), \tag{3.2}$$

where $a \wedge b$ is the smallest of a and b. The *inf* is still taken with respect to the set of trajectories y_x joining x to the boundary and satisfying:

$$\forall t \in [0, t_x] - y_x'(t) \in Dom(H^*(y_x(t), .).$$

x being supposed fixed, we can choose τ sufficiently small so that:

$$u(x) = \inf_{y_x}\{\int_0^{\tau} H^*(y_x(s), -y_x'(s))ds + u(y_x(\tau))\}. \tag{3.3}$$

To come up with our approximation scheme we use the following approximations:

$$\int_0^{\tau} H^*(y_x(s), -y_x'(s))ds \cong \tau\ H^*(y_x(0), -y_x'(0)),$$

$$y_x(\tau) \cong y_x(0) + \tau\ y_x'(0)$$

and

$$Dom(H^*(y_x(t), .) \cong Dom(H^*(y_x(0), .).$$

Since $y_x(0) = x$, equation (3.3) now becomes:

$$\sup_{-y_x'(0) \in Dom(H^*(x,.))} \{\frac{u(x) - u(x + \tau y_x'(0))}{\tau} - H^*(x, -y_x'(0))\} = 0; \tag{3.4}$$

Denoting $Dom(H^*(x, .))$ by D_x, we can simplify equation (3.4):

$$\sup_{-q \in D_x} \{\frac{u(x) - u(x + \tau q)}{\tau} - H^*(x, -q)\} = 0; \tag{3.5}$$

Remark:
The reader should note that equation (3.5) can be obtained directly by writing H as a supremum: $H(x,p) = \sup_{q \in Dom(H^*(x,.))}\{p.q - H^*(x,q)\}$.
In effect, since we want u to verify $H(x, \overrightarrow{\nabla u}(x)) = 0$ it must also verify

$$\sup_{q \in D_x} \{\overrightarrow{\nabla u}(x) \cdot q - H^*(x,q)\} = 0. \tag{3.6}$$

Interpreting $\overrightarrow{\nabla u}(x) \cdot q$ as a directional derivative, we can approximate $\overrightarrow{\nabla u}(x) \cdot q$ by

$$\overrightarrow{\nabla u}(x) \cdot q \quad \cong \quad \frac{u(x-\tau q) - u(x)}{-\tau}$$

and therefore equation (3.6) by:

$$\sup_{q \in D_x} \{ \frac{u(x-\tau q) - u(x)}{-\tau} - H^*(x,q) \} = 0$$

or

$$\sup_{-q \in D_x} \{ \frac{u(x) - u(x+\tau q)}{\tau} - H^*(x,-q) \} = 0.$$

Nevertheless this process is less legitimate than the first because nothing garantees that all functions u such that

$$\sup_{q \in D_x} \{ \overrightarrow{\nabla u}(x) \cdot q - H^*(x,q) \} = 0$$

are *viscosity solutions* of $H(x, \overrightarrow{\nabla u}(x)) = 0$, whereas there exist theorems which state (under some hypotheses) that a function verifying the dynamic programming principle (3.3) is a viscosity solution of this equation (see for example the theorem V.2.6 of [1]).

The final step consists of approximating u by a piecewise affine function in a neighbourhood of x (see figure 3.1) and calculating explicitly the *sup*. We now detail these two

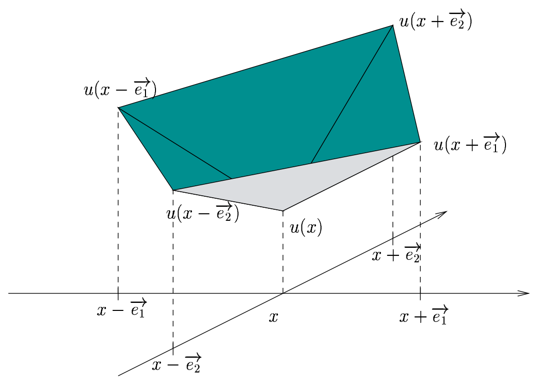


Figure 3.1: Piecewise affine approximation of u in a neighbourhood of x_{ij}

steps.

- Piecewise affine approximation of u:
  The piecewise affine approximation of u allows us to express $u(x+\tau q)$ as a weighted sum of $u(x)$, $u(x \pm \Delta x_1 \overrightarrow{e_1})$ and $u(x \pm \Delta x_2 \overrightarrow{e_2})$, where

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In order to express explicitly $u(x + \tau q)$, we need to partition the set D_x into four subsets D_1, D_2, D_3 and D_4 (see figure (3.2)),

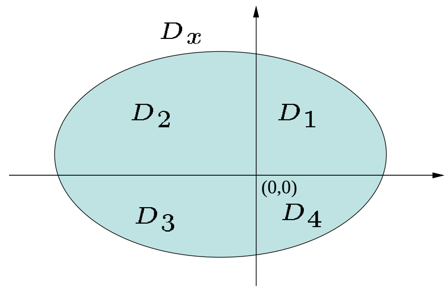


Figure 3.2: Partition of the set D_x into the four subsets D_1, D_2, D_3 and D_4: $D_x = D_1 \cup D_2 \cup D_3 \cup D_4$

$$\begin{array}{ll} D_1 = \{q \in D_x | q_x > 0, q_y \geq 0\}, & D_2 = \{q \in D_x | q_x \leq 0, q_y \geq 0\}, \\ D_3 = \{q \in D_x | q_x \leq 0, q_y < 0\}, & D_4 = \{q \in D_x | q_x > 0, q_y < 0\}. \end{array} \tag{3.7}$$

Let us detail the steps for D_1:
In this case we approximate $u(x + \tau q)$ with $u(x)$, $u(x + \Delta x_1 \overrightarrow{e_1})$ and $u(x + \Delta x_2 \overrightarrow{e_2})$. u being supposed to be affine, the affine coordinates of $u(x + \tau q)$ in the affine basis $(u(x),\ u(x + \Delta x_1 \overrightarrow{e_1}),\ u(x + \Delta x_2 \overrightarrow{e_2}))$ are the same as the affine coordinates of $x + \tau q$ in the affine basis $(x,\ x + \Delta x_1 \overrightarrow{e_1},\ x + \Delta x_2 \overrightarrow{e_2})$. By translation to the origin, these are also equal to the affine coordinates of τq in the affine basis $(0,\ \Delta x_1 \overrightarrow{e_1},\ \Delta x_2 \overrightarrow{e_2})$. For all q in D_1, let θ, μ and ν be the barycentric coordinates τq in the affine basis $(0,\ \Delta x_1 \overrightarrow{e_1},\ \Delta x_2 \overrightarrow{e_2})$:

$$\tau q = \theta.0 + \mu.\Delta x_1 e_1 + \nu.\Delta x_2 e_2.$$

Notice that $\theta = 1 - (\mu + \nu)$.
If we note $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ then

$$\mu = \frac{\tau q_1}{\Delta x_1} \quad \text{and} \quad \nu = \frac{\tau q_2}{\Delta x_2}.$$

We have

$$u(x + \tau q) = (1 - \frac{\tau q_1}{\Delta x_1} - \frac{\tau q_2}{\Delta x_2})\ u(x) + \frac{\tau q_1}{\Delta x_1}\ u(x + \Delta x_1 \overrightarrow{e_1}) + \frac{\tau q_2}{\Delta x_2}\ u(x + \Delta x_2 \overrightarrow{e_2}),$$

and

$$\frac{u(x) - u(x + \tau q)}{\tau} = q_1 \frac{u(x) - u(x + \Delta x_1 \overrightarrow{e_1})}{\Delta x_1} + q_2 \frac{u(x) - u(x + \Delta x_2 \overrightarrow{e_2})}{\Delta x_2}. \tag{3.8}$$

Let us note:

$$\begin{array}{rcl} a(\rho,x,t,u) & = & \frac{t-u(x-(\rho,0))}{\rho} \\ b(\rho,x,t,u) & = & \frac{t-u(x+(\rho,0))}{\rho} \\ c(\rho,x,t,u) & = & \frac{t-u(x-(0,\rho))}{\rho} \\ d(\rho,x,t,u) & = & \frac{t-u(x+(0,\rho))}{\rho}. \end{array} \tag{3.9}$$

So we can rewrite (3.8):

$$\frac{u(x)-u(x+\tau q)}{\tau} = q_1.b(\Delta x_1, x, u(x), u) + q_2.d(\Delta x_2, x, u(x), u).$$

The case of D_2, D_3 and D_4 are dealt with in the same way:

- for $q \in D_2$:

$$\frac{u(x)-u(x+\tau q)}{\tau} = -q_1.a(\Delta x_1, x, u(x), u) + q_2.d(\Delta x_2, x, u(x), u),$$

- for $q \in D_3$:

$$\frac{u(x)-u(x+\tau q)}{\tau} = -q_1.a(\Delta x_1, x, u(x), u) - q_2.c(\Delta x_2, x, u(x), u),$$

- for $q \in D_4$:

$$\frac{u(x)-u(x+\tau q)}{\tau} = q_1.b(\Delta x_1, x, u(x), u) - q_2.c(\Delta x_2, x, u(x), u).$$

Remark: It was foreseeable that the expressions $\frac{u(x)-u(x+\tau q)}{\tau}$ don't depend on τ.

- Explicit computation of the maximum:
  We need to maximize for $q \in -D_x$ the function:

$$K_x(q) = q \cdot \begin{pmatrix} A(\Delta x_1, x, u(x), u, q) \\ B(\Delta x_2, x, u(x), u, q) \end{pmatrix} - H^*(x,-q); \tag{3.10}$$

where

$$A(\Delta x_1, x, u(x), u, q) = \begin{cases} -a(\Delta x_1, x, u(x), u) & \text{if } q_1 \le 0 \\ b(\Delta x_1, x, u(x), u) & \text{if } q_1 > 0 \end{cases}$$

and

$$B(\Delta x_2, x, u(x), u, q) = \begin{cases} -c(\Delta x_2, x, u(x), u) & \text{if } q_2 \le 0 \\ d(\Delta x_2, x, u(x), u) & \text{if } q_2 > 0 \end{cases}$$

The maximization of (3.10) is performed independently over the four subsets D_1, D_2, D_3 and D_4. This step is the most technical of our method; in order to solve it we generally use the differential calculus.

In section 4.2 we deal fully with the example of the general equation of the SFS problem.

remark:
The reader will convince himself that the above method always yields an approximation scheme of the form $g(x, a, b, c, d) = 0$; with a, b, c and d given by (3.9).

3.2 Convergence of g(x,a,b,c,d) schemes with Dirichlet boundary conditions

In this section we are going to work in the framework of *discontinuous* viscosity solutions.

3.2.1 Approximation scheme: description

We specify here what we mean by $g(x, a, b, c, d)$ schemes with Dirichlet boundary conditions. Recall that we consider the following PDE with Dirichlet boundary conditions:

$$\begin{cases} H(x, u(x), \overrightarrow{\nabla u}(x)) = 0 & \forall x \in \Omega \\ \varphi(x) - u(x) = 0 & \forall x \in \partial\Omega \end{cases} \tag{3.11}$$

where Ω is a bounded open set of $\mathbb{R}^2$, H a continuous real function defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ and φ a continuous real function defined on $\partial\Omega$ (prolonged on a neighbourhood of $\partial\Omega$ by a continuous function renoted φ).
A "$g(x, a, b, c, d)$ scheme with Dirichlet boundary conditions" is an approximation scheme

$$S(\rho, x, u(x), u) = 0. \tag{3.12}$$

The locally bounded function $S : \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \times B(\overline{\Omega}) \longrightarrow \mathbb{R}$ is equal to:

$$S(\rho, x, t, u) = \begin{cases} g(x, a, b, c, d) & \text{if } x \in \Omega^\rho, \\ t - \varphi(x) & \text{if } x \in \flat\Omega^\rho \end{cases} \tag{3.13}$$

- a, b, c and d are functions of (ρ, x, t, u) defined by

$$\begin{array}{ll} a = \frac{t-u(x-(\rho,0))}{\rho} & b = \frac{t-u(x+(\rho,0))}{\rho} \\ c = \frac{t-u(x-(0,\rho))}{\rho} & d = \frac{t-u(x+(0,\rho))}{\rho}, \end{array} \tag{3.14}$$

and we have used the following notations:

- $\Omega^\rho = \{x \in \Omega | x - (\rho, 0), x + (\rho, 0), x - (0, \rho), x + (0, \rho) \in \overline{\Omega}\}$,
  and $\flat\Omega^\rho = \overline{\Omega} - \Omega^\rho$;
- g: $\Omega^\rho \times \mathbb{R}^4 \longrightarrow \mathbb{R}$,
- $B(D)$ is the set of real bounded functions defined on the set D.

3.2.2 Convergence towards viscosity solutions

We now give the definitions of *monotonicity*, *stability* and *consistence* of a numerical scheme according to Barles and Souganidis [3].

Definition 11 (monotonicity) *The scheme $S(\rho, x, u(x), u) = 0$ defined on $\overline{\Omega}$, is monotonous if*
$\forall \rho \in \mathbb{R}^+, \forall x \in \overline{\Omega}, \forall t \in \mathbb{R}$ *and* $\forall u, v \in B(\overline{\Omega})$,

$$u \leq v \quad \Longrightarrow \quad S(\rho, x, t, u) \geq S(\rho, x, t, v)$$

(that is to say: the scheme is nonincreasing with respect to u)

Definition 12 (stability) *The scheme $S(\rho, x, u(x), u) = 0$ defined on $\overline{\Omega}$, is stable if*

$$\forall \rho \in \mathbb{R}^{+*}, \text{ it has a solution } u^\rho \in B(\overline{\Omega}),$$

and if these solutions are bounded independently of ρ.

Recall that in the framework of discontinuous viscosity solutions, the equation (3.11) must be rewritten as:

$$F(x, u(x), \overrightarrow{\nabla u}(x)) = 0; \tag{3.15}$$

where F is defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ by

$$F(x, u, p) = \begin{cases} H(x, u, p) & \text{for } x \text{ in } \Omega, \\ u(x) - \varphi(x) & \text{for } x \text{ on } \partial\Omega. \end{cases} \tag{3.16}$$

Definition 13 (consistency) *The scheme $S(\rho, x, u^\rho(x), u^\rho) = 0$ defined on $\overline{\Omega}$, is consistent with equation (3.15) if :*
$\forall x \in \overline{\Omega}$ *et* $\forall \phi \in C_b^\infty(\overline{\Omega})$

$$\limsup_{\rho \to 0,\ y \to x,\ \xi \to 0} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \leq F^*(x, \phi(x), \nabla \phi(x)),$$

and

$$\liminf_{\rho \to 0,\ y \to x,\ \xi \to 0} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \geq F_*(x, \phi(x), \nabla \phi(x)).$$

With these definitions in hand we can now prove the following proposition:

Proposition 5 *With the above notations, if*

1. *g is nondecreasing with respect to each of the variables a, b, c and d,*
2. *for all ρ, there exists a bounded function u_0 such that $\forall x \in \overline{\Omega}$, $S(\rho, x, u_0(x), u_0) \leq 0$, and $\forall x \in \flat\Omega^\rho$, $u_0(x) = \varphi(x)$,*

3. *for all ρ, x and u, $\lim_{t\to+\infty} S(\rho,x,t,u) \geq 0$,*
4. *$g(x,a,b,c,d) \leq 0 \Longrightarrow a$ (b,c or d) is upper bounded independently of x,*
5. *g is continuous on $\Omega^\rho \times \mathbb{R}^4$ and $\forall x \in \Omega^\rho, \forall \phi \in C_b^\infty(\overline{\Omega})$*

$$g(x, \partial_{x_1}\phi(x), -\partial_{x_1}\phi(x), \partial_{x_2}\phi(x), -\partial_{x_2}\phi(x)) = H(x, \phi(x), \nabla\phi(x)),$$

then the scheme (3.12) is monotonous, stable and consistent with the equation (3.15).

△ Proof:

1. the monotonicity is clear by the hypothesis 1.
2. stability:
   The idea of the proof is the following:
   ρ being fixed, we construct a nondecreasing sequence of functions u_n such that for all x of $\overline{\Omega}$,
   $$S(\rho, x, u_n(x), u_n) \leq 0.$$
   We then show that this sequence is upper bounded therefore convergent. We note u^ρ its limit. For all x in $\overline{\Omega}$, the sequence $S(\rho, x, u_n(x), u_n)$ is periodically null. Thus by an argument of continuity, we prove that the limit u^ρ of the series u_n is the solution of the scheme (3.12).
   Let us now fix ρ.

   We first pave the set Ω^ρ with $\rho \times \rho$ squares (or subsets thereof), see figure (3.3). We order these squares lexicographically as $P^1, ..., P^q$; since Ω is bounded this is possible with a finite number of squares. It is important to note that the subsets $P^1, ..., P^q$ form a partition of Ω^ρ, i.e. that each $x \in \Omega^\rho$ belongs to one and only one P^i, $1 \leq i \leq q$. The partition is non unique.
   We define the following infinite periodic sequence $(P_n)_{n\in\mathbb{N}}$:
   $$P_n = P^i,$$
   where $i \equiv n\ [q]$.

   (a) Recursive construction of the sequence of functions $(u_n)_{n\in\mathbb{N}}$:
      i. Choose u_0 to be the function given by hypothesis 2.
      ii. Let us suppose that we have constructed the first n elements $(u_k)_{k=0..n-1}$ of our sequence such that:
      $$\forall k \in [0..n-2], \quad u_{k+1} \geq u_k,$$

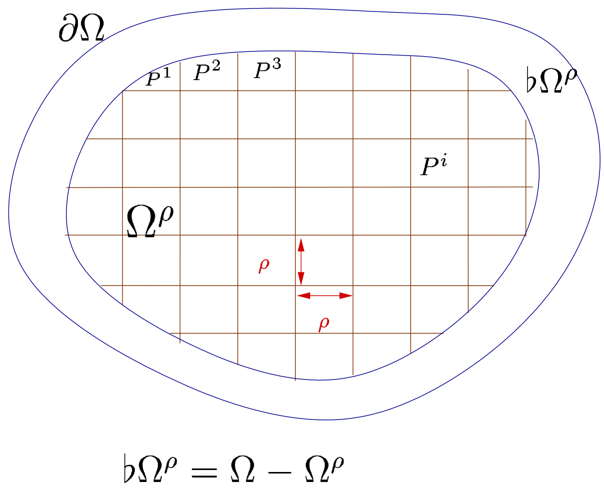


Figure 3.3: Partition of Ω^ρ .

$$\forall k \in [0..n-1], \forall x \in \flat\Omega^\rho, \quad u_k(x) = \varphi(x),$$

and $\forall k \in [0..n-1]$,

$$\forall x \in \overline{\Omega}, \quad S(\rho, x, u_k(x), u_k) \leq 0.$$

Using the square P_n,we now construct u_n:
$\forall x \in \overline{\Omega} - P_n$, let $u_n(x) = u_{n-1}(x)$
(this implies that $\forall x \in \flat\Omega^\rho,\ u_n(x) = \varphi(x)$);
$\forall x \in P_n,\ u_n(x)$ is chosen in such a way that

$$S(\rho, x, u_n(x), u_n) = 0. \tag{3.17}$$

Note that this is always possible:
$\triangle$ In effect, the recursivity hypothesis implies that

$$S(\rho, x, u_{n-1}(x), u_{n-1}) \leq 0;$$

Moreover, since $x \in P_n$ we have

$$x \pm \begin{pmatrix} \rho \\ 0 \end{pmatrix} \notin P_n \text{ and } x \pm \begin{pmatrix} 0 \\ \rho \end{pmatrix} \notin P_n,$$

therefore

$$u_n(x \pm \begin{pmatrix} \rho \\ 0 \end{pmatrix}) = u_{n-1}(x \pm \begin{pmatrix} \rho \\ 0 \end{pmatrix}),$$

and

$$u_n(x \pm \begin{pmatrix} 0 \\ \rho \end{pmatrix}) = u_{n-1}(x \pm \begin{pmatrix} 0 \\ \rho \end{pmatrix}).$$

By (3.13) and (3.14), we obtain:

$$\forall x \in P_n, \quad S(\rho, x, u_{n-1}(x), u_n) = S(\rho, x, u_{n-1}(x), u_{n-1}),$$

and hence

$$\forall x \in P_n, \quad S(\rho, x, u_{n-1}(x), u_n) \leq 0. \tag{3.18}$$

At last, hypothesis 3 implies

$$\lim_{t \to +\infty} S(\rho, x, t, u_n) \geq 0.$$

The "intermediate values" theorem applied to the continuous function (hypothesis 5)

$$t \mapsto S(\rho, x, t, u_n)$$

allows us to conclude.
□

- Let us notice that hypothesis 1 guarantees that the function $t \mapsto S(\rho, x, t, u_n)$ is nondecreasing; then by (3.17) and (3.18), for all x in P_n we have $u_n(x) \geq u_{n-1}(x)$. Since $U_n(x) = U_{n-1}(x)$ $\forall x \in \overline{\Omega} - P_n$ we have

$$u_n \geq u_{n-1} \text{ on } \overline{\Omega}.$$

- Finally, we have
  - ◇ $\forall x \in \Omega^\rho - P_n, \quad S(\rho, x, u_n(x), u_n) \leq 0$
    because

$$\begin{aligned} S(\rho, x, u_n(x), u_n) &= S(\rho, x, u_{n-1}(x), u_n) \\ &\quad \downarrow \text{ [monotonicity of the scheme]} \\ &\leq S(\rho, x, u_{n-1}(x), u_{n-1}) \\ &\quad \downarrow \text{[recursivity hypothesis]} \\ &\leq 0 \end{aligned}$$

  - ◇ $\forall x \in \flat\Omega^\rho$,

$$\begin{aligned} S(\rho, x, u_n(x), u_n) &= u_n(x) - \varphi(x), \\ &= 0. \end{aligned}$$

Thus we have constructed a nondecreasing sequence of functions u_n defined on $\overline{\Omega}$, such that $\forall x \in \overline{\Omega}, S(\rho, x, u_n(x), u_n) \leq 0$.
For all x in $\flat\Omega^\rho$ the sequence $S(\rho, x, u_n(x), u_n)$ is equal to 0. For all x in Ω^ρ, there exists $i \in [1..q]$ such that $x \in P^i$ and hence $S(\rho, x, u_n(x), u_n) = 0$ for all n in the set $\{i + mq; m \in \mathbb{N}\}$.

(b) The sequence u_n is bounded (independently of x):
- $\forall x \in \flat\Omega^\rho$, $u_n(x) = \varphi(x)$; therefore u_n is bounded on $\flat\Omega^\rho$.
- $\forall x \in \Omega^\rho$:
  We have proved that

$$\forall n, \ \forall y \in \Omega^\rho, \ S(\rho, y, u_n(y), u_n) \leq 0.$$

Because of hypothesis 4, $\exists M \ / \ \forall y \in \Omega^\rho$,

$$a(\rho, y, u_n(y), u_n) \leq M.$$

Hence

$$\frac{u_n(y) - u_n(y - \begin{pmatrix} \rho \\ 0 \end{pmatrix})}{\rho} \leq M,$$

from which one concludes

$$u_n(y) \leq u_n(y - \begin{pmatrix} \rho \\ 0 \end{pmatrix}) + \rho M.$$

Then we have by recurrence:

$$u_n(x) \leq XM + \max_{z \in \flat\Omega^\rho} \varphi(z).$$

(X is the horizontal width of $\overline{\Omega}$). The sequence u_n is therefore upper bounded independently of x.

(c) Convergence and properties of the limit:
Being nondecreasing and upper bounded, the sequence u_n converges toward a limit. We note this limit u^ρ.
Let us fix x in Ω^ρ and define:

$$a_n = \frac{u_n(x) - u_n(x - (\rho, 0))}{\rho}, \quad b_n = \frac{u_n(x) - u_n(x + (\rho, 0))}{\rho},$$

$$c_n = \frac{u_n(x) - u_n(x - (0, \rho))}{\rho}, \quad d_n = \frac{u_n(x) - u_n(x + (0, \rho))}{\rho};$$

and

$$a^\rho = \frac{u^\rho(x) - u^\rho(x - (\rho, 0))}{\rho}, \quad b^\rho = \frac{u^\rho(x) - u^\rho(x + (\rho, 0))}{\rho},$$

$$c^\rho = \frac{u^\rho(x) - u^\rho(x - (0, \rho))}{\rho}, \quad d^\rho = \frac{u^\rho(x) - u^\rho(x + (0, \rho))}{\rho}.$$

u_n converging toward u^ρ, a_n, b_n, c_n and d_n converge toward a^ρ, b^ρ, c^ρ and d^ρ, respectively. By continuity of g (hypothesis 5),

$$g(x, a_n, b_n, c_n, d_n) \longrightarrow g(x, a^\rho, b^\rho, c^\rho, d^\rho);$$

in other words, the sequence $\left(\ S(\rho, x, u_n(x), u_n) \ \right)_{n\in\mathbb{N}}$ converges to $S(\rho, x, u^\rho(x), u^\rho)$. Since $\left(\ S(\rho, x, u_n(x), u_n) \ \right)_{n\in\mathbb{N}}$ is periodically null,

$$\forall x \in \overline{\Omega}, \quad S(\rho, x, u^\rho(x), u^\rho) = 0,$$

and u^ρ is the solution of our scheme.
Let us notice that

$$u^\rho \leq M \ X + \max_{z\in\flat\Omega^\rho} \varphi(z)$$

(independently of ρ).

3. Consistency.
Instead of considering the scheme

$$S(\rho, x, u(x), u) = 0 \qquad \text{in } \overline{\Omega},$$

we are going to consider the scheme

$$S'(\rho, x, u(x), u) = 0 \qquad \text{in } \overline{\Omega},$$

such that

$$S'(\rho, x, t, u) = \rho S(\rho, x, t, u).$$

Monotonicity, stability and solutions are exactly the same for both schemes.
We prove that S' is consistent with equation (3.15), i.e.:
$\forall x \in \overline{\Omega}$ et $\forall \phi \in C_b^\infty(\overline{\Omega})$

i)

$$\limsup_{\rho\to 0, \ y\to x, \ \xi\to 0} \frac{S'(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \leq F^*(x, \phi(x), \nabla\phi(x)).$$

ii)

$$\liminf_{\rho\to 0, \ y\to x, \ \xi\to 0} \frac{S'(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \geq F_*(x, \phi(x), \nabla\phi(x)).$$

We prove i):

(a) if $x \in \Omega$: For ρ sufficiently small and y sufficiently close to x, we have $y \in \Omega^\rho$, and

$$\frac{S'(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} = g(y, a, b, c, d)$$

where a, b, c and d are evaluated at $(\rho, y, \phi(y)+\xi, \phi+\xi)$. Using the "finite increases" theorem and the fact that the derivative of ϕ is continuous, we have:

$$\lim_{\rho\to 0, \ y\to x} a(\rho, y, \phi(y) + \xi, \phi + \xi) = \partial_x\phi(x),$$

$$\lim_{\rho\to 0,\ y\to x} b(\rho, y, \phi(y)+\xi, \phi+\xi) = -\partial_x\phi(x),$$
$$\lim_{\rho\to 0,\ y\to x} c(\rho, y, \phi(y)+\xi, \phi+\xi) = \partial_y\phi(x),$$
$$\lim_{\rho\to 0,\ y\to x} d(\rho, y, \phi(y)+\xi, \phi+\xi) = -\partial_y\phi(x),$$

By continuity of g (hypothesis 5), we have:

$$\limsup_{\rho\to 0,\ y\to x,\ \xi\to 0} \frac{S'(\rho, y, \phi(y)+\xi, \phi+\xi)}{\rho}$$
$$= g(x, \partial_x\phi(x), -\partial_x\phi(x), \partial_y\phi(x), -\partial_y\phi(x))$$
$$= H(x, \phi(x), \nabla\phi(x)) \text{ (by hypothesis 5)}$$
$$= F^*(x, \phi(x), \nabla\phi(x)) \text{ (because } x\in\Omega \text{ and H is continuous).}$$

(b) if $x \in \partial\Omega$, we have also by continuity of the considered functions:

$$\limsup_{\rho\to 0,\ y\to x,\ \xi\to 0} \frac{S'(\rho, y, \phi(y)+\xi, \phi+\xi)}{\rho}$$
$$= \max(g(x, \partial_x\phi(x), -\partial_x\phi(x), \partial_y\phi(x), -\partial_y\phi(x)), \varphi(x) - \phi(x))$$
$$= \max(H(x, \phi(x), \nabla\phi(x)), \varphi(x) - \phi(x));$$
$$= F^*(x, \phi(x), \nabla\phi(x)).$$

We deal the same way with point ii), comparing the inferior limit and F_*.

Thus, the scheme S' is consistent with the equation (3.15); This finishes the proof of proposition 5.
□

We now recall the definitions of the functions $\overline{u}$ and $\underline{u}$:

Definition 14 $\forall x \in \overline{\Omega}$,

$$\overline{u}(x) = \limsup_{\substack{\rho\to 0,\\ y\to x,\, y \text{ in } \overline{\Omega}}} u^\rho(y)$$
$$\underline{u}(x) = \liminf_{\substack{\rho\to 0,\\ y\to x,\, y \text{ in } \overline{\Omega}}} u^\rho(y)$$

With the results in the article [3] of Barles and Souganidis, we can state the following theorem:

Theorem 7 *If the approximation scheme S is monotonous, stable and consistent with equation (3.15) then $\overline{u}$ and $\underline{u}$ are respectively viscosity subsolution and supersolution of this equation.*

$\triangle$ Proof: see theorem 2.1 of [3].
$\square$
We finally prove the essential result of this section:

Theorem 8 *Suppose that the hypotheses of proposition 5 and theorem 4 (strong uniqueness) are verified. Then the solution u^ρ of the scheme (3.12) converges toward a viscosity solution of (3.15) when $\rho \to 0$.*

The proof is based on theorems 4 and 7. The idea is the following:

1. we apply theorem 7 to prove that $\overline{u}$ and $\underline{u}$ are viscosity subsolution and supersolution of our equation, respectively.
2. we apply theorem 4 to conclude that $\overline{u} = \underline{u}$ on Ω.

$\triangle$ Proof:
The hypotheses of proposition 5 being verified, we have the monotonicity, the stability and the consistency (with the equation) of our scheme. By theorem 7, the functions $\overline{u}$ and $\underline{u}$ are respectively viscosity subsolution and supersolution of the PDE (3.15). The hypotheses of theorem 4 being also verified, $\overline{u} \leq \underline{u}$ on Ω. By definition, we have $\underline{u} \leq \overline{u}$, and hence:

$$\overline{u} = \underline{u} \quad (\text{ on } \Omega).$$

For all $x \in \Omega$, we now know that the limit

$$\lim_{y \to x, \rho \to 0} u^\rho(y),$$

and a fortiori the limit

$$\lim_{\rho \to 0} u^\rho(x)$$

exist and are equal to $\overline{u}(x)$ and $\underline{u}(x)$. The limit $u := \lim_{\rho \to 0} u^\rho$ defined on Ω, is then both l.s.c and u.s.c; It is therefore continuous.
Finally, let v be a viscosity solution of (3.15). By definition 6, v^* and v_* is a subsolution and a supersolution, respectively; by theorem 4 we have:

$$\forall x \in \Omega, \overline{u}(x) \leq v_*(x) \leq v^*(x) \leq \underline{u}(x)$$

(see definition 5 for the definition of v_* and v^*).
Therefore

$$u = v \quad \text{ on } \Omega.$$

At last, $\forall x \in \partial\Omega$, since $\forall \rho,\ u^\rho(x) = \varphi(x)$, the limit

$$u(x) := \lim_{\rho \to 0} u^\rho(x)$$

exists and is equal to $\varphi(x)$.
We then have

$$u^\rho \to u \text{ on } \overline{\Omega},$$

where u is equal to the viscosity solution on Ω, and to φ on $\partial\Omega$.

Let us remark that u is continuous on Ω but not on $\overline{\Omega}$.
□

3.3 Construction of numerical algorithms and proof of convergence

In the previous sections of this chapter, we have given a systematic method allowing to obtain a numerical scheme for a Hamilton-Jacobi-Bellman equation. Thanks to theorem 8, we have proved that when $\rho = (\Delta x_1, \Delta x_2)$ tends to zero, the solution u^ρ of the numerical scheme converges to the unique viscosity solution of the corresponding equation. We are now going to describe a systematic method for constructing an algorithm that computes an approximation of u^ρ, for each value of $\rho > 0$. We also prove the convergence of our algorithm. It is important to keep in mind that this algorithm converges toward u^ρ but not toward the viscosity solution.

For a fixed $\rho = (\Delta x_1, \Delta x_2)$, let us note

- $x_{ij} = \begin{pmatrix} i\Delta x_1 \\ j\Delta x_2 \end{pmatrix}$ for (i,j) in $\mathbb{Z}^2$,
- $\mathcal{X} := \{x_{ij}\}_{(i,j)\in\mathbb{Z}^2}$,
- Q^ρ and Q the intersections of the discrete mesh $\mathcal{X}$ with Ω^ρ and $\overline{\Omega}$, respectively. In other words:

$$Q^\rho = \Omega^\rho \cap \mathcal{X},$$
$$Q = \overline{\Omega} \cap \mathcal{X}.$$

The following algorithm computes the sequence of approximations U_{ij}^n of $u^\rho(x_{ij})$, and that for all (i,j) such as $x_{ij} \in Q$:

Algorithm 1 *1. Initialisation* $(n = 0)$*:* $U_{ij}^0 = u_0(x_{ij})$.

2. *Choice of a pixel $x_{ij} \in Q^\rho$ and modification (step $n+1$) of U^n_{ij}:*
   We choose $U^{n+1} = \sup\{V = (V_{k,l})_{x_{kl} \in Q}$ such that

$$\forall (k,l) \neq (i,j), \quad V_{kl} = U^n_{kl} \ \text{ and } \ S(\rho, x_{ij}, V_{ij}, V) = 0\}.$$

3. *Choose the next pixel $x_{ij} \in Q^\rho$ in such a way that all pixels of Q^ρ are regularly visited and go back to 2.*

Remark: u_0 must verify hypothesis 2 of proposition 5.

Definition 15 *The algorithm 1 is* well-defined *if for all steps, the set of V's defined at step (2) of the algorithm, is not empty and bounded; in other words, if for all steps we can compute the next approximation.*

We have the following theorem:

Theorem 9 *If the hypotheses of proposition 5 are satisfied, the algorithm 1 is well-defined and the constructed sequence U^n is increasing and converges when $n \longrightarrow +\infty$ towards the solution u^ρ of the scheme (3.12).*

△ Proof:
The elements of the sequence U_n are the restrictions to Q of the functions u_n introduced in the step 2 ("stability") of the proof of proposition 5.
□

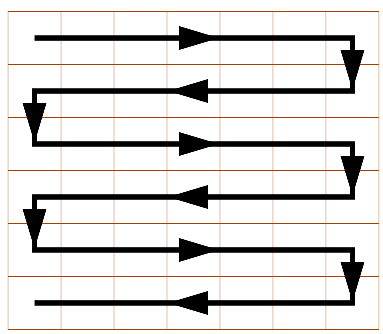


Figure 3.4: Path proposed by R.Kimmel

Let us emphasize the fact that the limit does not depend on the particular path used to traverse the pixels. Nevertheless, the convergence velocity strongly depends on this choice. For example, the strategy (proposed by R.Kimmel[1]) which consists in following back and forth the path indicated in figure 3.4 is the most effective one we have tested so far.

[1] personal communication...

3.4 Other tools proposed in literature

The reader should know that other methods can be founded in the literature. For example, the Applied Mathematics group in the Roma university ("La Sapienza") proposes approximation schemes based on finite elements. The interested reader is referred to the recent book edited by M.Falcone and C.Makridakis [15] , to the chapter VI and appendix A of [1], and to the references there in.

Chapter 4

Application to the Shape-from-Shading problem

4.1 Application of the theorems given in the previous chapters

In this section, we are going to show that the theorems presented in the chapter 2 apply to the shape from shading equations. Recall that the brightness of the photography at point x verifies the irradiance equation (see chapter 1):

$$\forall x \in \Omega, \quad I(x) = \frac{-\overrightarrow{\nabla u}(x) \cdot \mathbf{l} + \gamma}{\sqrt{1 + |\overrightarrow{\nabla u}(x)|^2}}.$$

Different Hamiltonians can be deduced from the above irradiance equation; since our theory applies only with *convex* (with respect to p) Hamiltonians, we choose the following ones:

- in the case where $\mathbf{L} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

$$H(x,p) = |p| - \sqrt{\frac{1}{I(x)^2} - 1}; \tag{4.1}$$

- in the general case:

$$H(x,p) = I(x)\sqrt{1 + |p|^2} + p \cdot \mathbf{l} - \gamma. \tag{4.2}$$

Let us not forget that in order to have a well-posed problem, we have to add boundary conditions; Thus the equations to consider are:

$$\begin{cases} |\overrightarrow{\nabla u}(x)| - \sqrt{\frac{1}{I(x)^2} - 1} = 0 & \forall x \in \Omega, \\ u(x) = \varphi(x) & \forall x \in \partial\Omega; \end{cases} \tag{4.3}$$

and

$$\begin{cases} I(x)\sqrt{1 + |\overrightarrow{\nabla u}(x)|^2} + \overrightarrow{\nabla u}(x) \cdot \mathbf{l} - \gamma = 0 & \forall x \in \Omega, \\ u(x) = \varphi(x) & \forall x \in \partial\Omega. \end{cases} \tag{4.4}$$

Remarks:

1. In this chapter we will only work with the general equation (4.4); we will not deal with the Eikonal equation (4.3). Most of the results regarding the Eikonal equation can be found in E.Rouy-A.Tourin's article [29]. A very good exercise for the interested reader is to apply the tools and methods presented here to the Eikonal equation. The reader will *easily* find again the results of [29] (existence, uniqueness, scheme, algorithm).

2. The reader has most probably noticed that all theorems presented in chapter 2 impose regularity with respect to the space variable x to Hamiltonians. These regularity hypotheses impose to take intensity functions I which are Lipschitz continuous. Consequently the following results are only valid with *continuous* image I !
   Most probably, recent work will allow shortly to extend the theory to discontinuous Hamiltonians in the space variables (see [24], [27] and the work of A. Siconolfi and F. Camilli[1])

From now on, we suppose I *Lipschitz continuous.*

4.1.1 Uniqueness results

In the general case, where the Hamiltonian H is

$$H(x,p) = I(x)\sqrt{1 + |p|^2} + p \cdot \mathbf{l} - \gamma, \tag{4.5}$$

we have:

1. Since I is lipschitz continuous, all hypotheses of regularity on H with respect to x are verified for the theorems 2 and 4.

2. Since $\forall x \in \Omega$, $H(x,\cdot) : p \mapsto H(x,p)$ is $C^1(\mathbb{R}^2)$ with $|\overrightarrow{\nabla H(x,\cdot)}(p)| \leq 2$, then H is Lipschitz with respect to p independently of x.

[1]Talk at the TMR conference entitled "Eikonal equations with measurable dependence on the state variable" (6th-8th march 2002, in Tours, France); article to appear.

3. H is convex with respect to p.

4. Strict viscosity subsolution:
   It is easy to verify that if I does not reach the value 1 within Ω, that is to say, if

$$\forall x \in \Omega, \quad 0 \leq I(x) < 1,$$

   then the function

$$\tilde{u} : (x, y) \longmapsto \frac{1}{\gamma}(-\alpha x - \beta y)$$

   is a strict viscosity subsolution of (4.5).

Thus, as soon as $I < 1$ on Ω, all hypotheses of the theorem 2 are verified. This implies the uniqueness of continuous solutions.

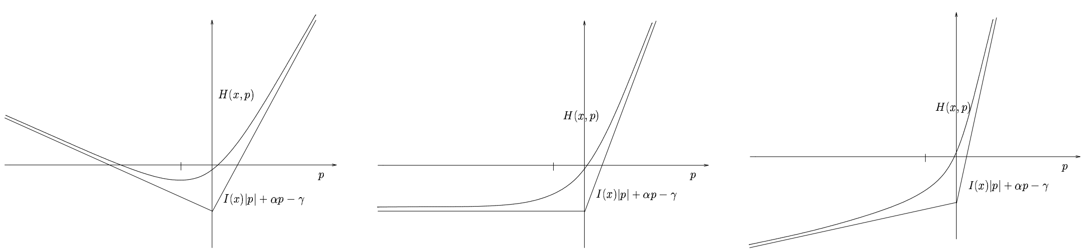
What about theorem 4 which deals with discontinuous solutions :
As above, it is easy to prove that $0 \leq I < 1$ on $\overline{\Omega}$ validates the "HNCL" Hypotheses.
The other two hypotheses of theorem 4

- $\forall x \in \partial\Omega, \quad H(x, p + \lambda\eta(x)) \leq 0 \quad \Longrightarrow \quad \lambda \leq C(1 + |p|)$,
- $H(x, p - \lambda\eta(x)) \longrightarrow +\infty$ uniformly with respect to x, when $\lambda \to +\infty$;

are verified if and only if

$$\forall x \in \overline{\Omega}, \quad I(x) > |\mathbf{l}|$$

(see figure (4.1) representing H and see the remarks of the section (2.2.3)).
Then $|\mathbf{l}| < I < 1$ on $\overline{\Omega}$ implies the strong uniqueness of the discontinuous solutions on Ω (theorem 4).
Recall that we do not have uniqueness on $\overline{\Omega}$; but in practice we are only interested in the values of the solution on Ω. On $\partial\Omega$, we will take the values given by the boundary condition φ.



a) $I(x) > |\mathbf{l}|$ b) $I(x) = |\mathbf{l}|$ c) $I(x) < |\mathbf{l}|$

Figure 4.1: *1D* description of H (with $\beta = 0$) when $I(x) < |\mathbf{l}|$, $I(x) = |\mathbf{l}|$ and $I(x) < |\mathbf{l}|$

In practice, I can reach the value 1 in an arbitrary compact set in $\overline{\Omega}$. This implies that there does not exist a strict viscosity subsolution and we lose uniqueness. The loss of uniqueness is completely characterized in [26] and we summarize it here. We assume that there exists a finite collection of disjoint connected compact sets $K_i, i = 1, \cdots, n$ such that $\{I = 1\} = \cup_{i=1}^{n} K_i$. The main points are that the viscosity solutions of (4.4) are constant over the K_i's and, when $n > 1$, we only need to specify the differences of the values of u in the K_i's. This allows us to ignore the set $\{I = 1\}$ and to work in the open set $\Omega' = \Omega - \{I = 1\}$. In other words, when the set $\{I = 1\}$ is not empty we consider the problem

$$\begin{cases} I(x)\sqrt{1 + |\overrightarrow{\nabla u}(x)|^2} + \overrightarrow{\nabla u}(x) \cdot \mathbf{l} - \gamma = 0 & \forall x \in \Omega' \\ u(x) = \varphi(x) & \forall x \in \partial\Omega' \end{cases}, \tag{4.6}$$

rather than (4.4). Thus, in the continuous case, as soon as I is Lipschitz continuous, we have a uniqueness result for problem (4.6). The solutions of (4.4) are then obtained from these by choosing (almost) arbitrary values for u in the K_i's. Another possibility is to choose among all solutions one which possesses an extra property, as in the work of M. Falcone and al. [13, 14] where the uniqueness is obtained by choosing the *maximal* solution.

4.1.2 Existence results

In this section we are going to apply theorem 3 (to find continuous solutions) and theorem 5 (to find discontinuous solutions) with the general equation (4.4).

- The continuous case:
  Since I is continuous, H is continuous in $\overline{\Omega} \times \mathbb{R}^2$ and convex with respect to p; we easily obtain, taking the derivative:

$$\inf_{p \in \mathbb{R}^2} H(x,p) = \begin{cases} \sqrt{I^2(x) - |\mathbf{l}|^2} - \gamma & \text{if } I(x)^2 \geq \alpha^2 + \beta^2. \\ -\infty & \text{otherwise.} \end{cases}$$

  Since $I^2 \leq \alpha^2 + \beta^2 + \gamma^2 = 1$, we have $\inf_{p \in \mathbb{R}^2} H(x,p) \leq 0$. Finally, since Ω is bounded hypothesis 2 in theorem 3 is satisfied iff $I > |\mathbf{l}|$ and, if the compatibility condition (2.10) is satisfied on $\partial\Omega$, we have obtained the existence of a continuous viscosity solution of problem (4.4).

- The discontinuous case:
  As we know, to apply theorem 5 we need to express H as a supremum:

$$H(x,a) = \sup_{a \in A} \{-f(x,a) \cdot p - l(x,a)\}$$

  In the section (2.2.4), we have described a method for obtaining this expression (recall that H has to be convex!) and we have detailed an example: this example dealt exactly with our Hamiltonian (4.2).
  So we have:

$$f(x,q) = -I(x) \cdot q - \mathbf{l},$$

$$l(x,q) = \gamma - I(x)\sqrt{1-|q|^2},$$
$$A = \overline{B}(0,1).$$

The reader can verify that theorem 5 applies as soon as I is Lipschitz in $\overline{\Omega}$ (and even if $I^2 < \alpha^2 + \beta^2$ or $I = 0$!).

Finally, in both cases a solution is given by

$$u(x) = \inf_{\xi}\{\int_0^{T_0} H^*(\xi(s), -\xi'(s))\, ds + \varphi(\xi(T_0))\}, \tag{4.7}$$

where $\xi \in \cup_{y\in\partial\Omega} C_{x,y}$ satisfies $\forall t \in [0,T_0]$, $-\xi'(t) \in \overline{B}(\mathbf{l}, I(\xi(t)))$ and $C_{x,y}$ is defined on page 16.
Recall that:

$$H^*(x,q) = \begin{cases} -\sqrt{I(x)^2 - |q-\mathbf{l}|^2} + \gamma & \text{if } |q-\mathbf{l}| \le I(x) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

In summary:
If $I > |\,\mathbf{l}\,|$ and the compatibility condition[2] is verified then there exist *continuous* solutions. Otherwise, for all I, there always exist *discontinous* solutions and in all cases the function u given by (4.7) and (4.8) is one.

4.1.3 Stability results

In computer vision or more generally in image processing, the images are always corrupted by noise. It is therefore very important to design schemes and algorithms *robust* to noise. That is to say we would like that the result obtained by the algorithm from a noisy image be close to the ideal result obtained from the perfect image. This property is often difficult to guarantee.

For the "SFS" problem, the robustness is mathematically expressed by the continuity of the application which, given an image I, returns the associated surface u. In other words, we would like that, for all sequences of noisy images I_ϵ uniformly converging toward an image I, the sequence of recovered solutions u_ϵ uniformly converges toward the solution u associated to I.

Let $I : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous image verifying $|\mathbf{l}| < I < 1$.
Let $(I_\epsilon)_{\epsilon>0}$ be a sequence of (Lipschitz continuous with the same Lipschitz constant) noisy images such that

$$I_\epsilon \longrightarrow I \text{ when } \epsilon \to 0$$

[2] see the remark on page 17.

uniformly on $\overline{\Omega}$.
Let φ be a continuous function defined on $\partial\Omega$.
Let us note

$$H(x,p) = I(x)\sqrt{1+|p|^2} + p\cdot \mathbf{l} - \gamma, \tag{4.9}$$

$$H_\epsilon(x,p) = I_\epsilon(x)\sqrt{1+|p|^2} + p\cdot \mathbf{l} - \gamma, \tag{4.10}$$

$$F(x,u,p) = \begin{cases} H(x,p) & \forall x \in \Omega, \\ u - \varphi(x) & \forall x \in \partial\Omega; \end{cases} \tag{4.11}$$

and

$$F_\epsilon(x,u,p) = \begin{cases} H_\epsilon(x,p) & \forall x \in \Omega, \\ u - \varphi(x) & \forall x \in \partial\Omega. \end{cases} \tag{4.12}$$

Since $\overline{\Omega}$ is compact, there exists a $\delta > 0$ such that:

$$\forall x \in \overline{\Omega}, \qquad |\mathbf{l}| + \delta < I(x) < 1 - \delta$$

and

$$\forall x \in \overline{\Omega}, \qquad |\mathbf{l}| + \delta < I_\epsilon(x) < 1 - \delta.$$

I and I_ϵ being Lipschitz continuous, according to sections 4.1.1 we can claim that the strong uniqueness property is true for the Hamiltonians H and H_ϵ. Also the equations

$$(E) \qquad\qquad F(x,u(x),\overrightarrow{\nabla u}(x)) = 0 \quad \text{on } \overline{\Omega},$$

and

$$(E_\epsilon) \qquad\qquad F_\epsilon(x,u(x),\overrightarrow{\nabla u}(x)) = 0 \quad \text{on } \overline{\Omega},$$

have discontinuous viscosity solutions (unique on Ω). We note u and u_ϵ the solutions of the equations (E) and (E_ϵ), respectively, given by the formula (4.7) applied with the adequate Hamiltonian. Let us recall that, by the strong uniqueness property, u and u_ϵ are continuous on Ω.

Proposition 6 *With the above notations and hypotheses,*

$$u_\epsilon \longrightarrow u \text{ when } \epsilon \to 0$$

uniformly on Ω.

$\triangle$ Proof:
We verify the hypotheses of corollary 2.

1. Since H and φ are continuous on $\overline{\Omega} \times \mathbb{R}^2$ and $\partial\Omega$ respectively, *F is locally bounded.* For all x in Ω, $F(x,u,p) = H(x,p)$. Therefore *F verifies the strong uniqueness property.*

2. $(F_\epsilon)_{\epsilon>0}$ *converges locally uniformly toward* F.

   Let (x,u,p) be in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2$ and V be a bounded neighbourhood of (x,u,p) in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2$.
   V being bounded we can consider the maximum

   $$M := \max_{(y,v,q)\in V} \sqrt{1+|q|^2}.$$

   Since $I_\epsilon \longrightarrow I$ when $\epsilon\to 0$ uniformly on $\overline{\Omega}$, then

   $\forall\delta>0,\ \exists V_0$ neighbourhood of 0 such that $\forall\epsilon\in V_0$ and $\forall(y,v,q)\in V \quad |I(y)-I_\epsilon(y)|\leq\delta$.

   Thus we can upper-bound $|F_\epsilon(y,v,q)-F(y,v,q)|$:

   ∘ if $y\in\Omega$, by formulas (4.11) and (4.12),

   $$|F_\epsilon(y,v,q)-F(y,v,q)| = |H_\epsilon(y,q)-H(y,q)|;$$

   by formulas (4.9) and (4.10), we have

   $$|H_\epsilon(y,q)-H(y,q)| \leq |I_\epsilon(y)-I(y)|\sqrt{1+|q|^2}$$

   then

   $$|F_\epsilon(y,v,q)-F(y,v,q)| \leq M\delta.$$

   ∘ if $y\in\partial\Omega$, clearly

   $$|F_\epsilon(y,v,q)-F(y,v,q)| = |(v-\varphi(y)) \quad - \quad (v-\varphi(y))| \quad \leq \quad M\delta.$$

   We conclude that $\forall\delta>0,\ \exists V_0$ neighbourhood of 0 such that $\forall\epsilon\in V_0$ and $\forall(y,v,q)\in V$

   $$|F_\epsilon(y,v,q)-F(y,v,q)| \leq \delta.$$

3. *The sequence* $(u_\epsilon)_\epsilon$ *is locally uniformly bounded on* $\overline{\Omega}$.

   ∘ We need the lemmas 2 and 3:

   Lemma 2 *Let I_1 and I_2 be two Lipschitz continuous real functions defined on $\overline{\Omega}$ such that there exists $\delta>0$ such that $|1|+\delta<I_i<1-\delta$ $(i=1,2)$. Let u_1 and u_2 be the solutions of the equations associated to I_1 and I_2, given by (4.7). If*

   $$I_1 \geq I_2 \text{ on } \overline{\Omega}$$

   then

   $$u_1 \geq u_2 \text{ on } \overline{\Omega}.$$

$\triangle$ Proof:
The reader will easily verify this lemma. $\square$

Lemma 3 *Let λ be in $\mathbb{R}^{+*}$.*
Let $(I_\epsilon)_{\epsilon>0}$ be a sequence of λ-Lipschitz continuous real functions on a normed vector space $(E, |\ .\ |)$. The functions I_{min} and I_{max} defined by

$$I_{min}(x) = \inf_{\epsilon>0} I_\epsilon(x)$$

$$I_{max}(x) = \sup_{\epsilon>0} I_\epsilon(x)$$

are λ-Lipschitz continuous.

$\triangle$ Proof:
Let x and y be in E.
For all $\epsilon > 0$, since the functions I_ϵ are λ-Lipschitz continuous,

$$I_\epsilon(y) \leq I_\epsilon(x) + \lambda|x - y|.$$

By definition of I_{max},

$$I_\epsilon(x) \leq I_{max}(x),$$

therefore

$$I_\epsilon(y) \leq I_{max}(x) + \lambda|x - y|.$$

Thus, taking the sup with respect to ϵ, we have

$$I_{max}(y) \leq I_{max}(x) + \lambda|x - y|.$$

Exchanging the roles of x and y, we obtain

$$|I_{max}(x) - I_{max}(y)| \leq \lambda|x - y|.$$

We do the same with I_{min}.
$\square$

- Let I_{min} and I_{max} be the new images defined by

$$I_{min} = \inf_{\epsilon>0} I_\epsilon,$$

and

$$I_{max} = \sup_{\epsilon>0} I_\epsilon.$$

By lemma 3, I_{min} and I_{max} are Lipschitz continuous. Also we have

$$|\mathbf{l}| + \delta \leq I_{min} \leq 1 - \delta,$$

and

$$|1| + \delta \leq I_{max} \leq 1 - \delta.$$

Then, like for the images I and I_ϵ, the formula (4.7) gives solutions of the associated equations to the image I_{min} and I_{max}. We note these solutions u_{min} and u_{max}.
Remark: by definition, u_{min} and u_{max} are locally bounded on the compact $\overline{\Omega}$, hence bounded on $\overline{\Omega}$.
Since for all $\epsilon > 0$

$$I_{min} < I_\epsilon < I_{max},$$

lemma 2 implies

$$u_{min} < u_\epsilon < u_{max}.$$

Thus the functions u_ϵ are uniformly bounded on $\overline{\Omega}$.

4. Applying corollary 2 and proposition 4, we conclude that the sequence $(u_\epsilon)_{\epsilon>0}$ uniformly converges toward u on all compact of Ω.

□

4.2 The associated scheme

4.2.1 Explicit calculation of the scheme

In this section, we find the scheme associated to the equation

$$\begin{cases} H(x, \overrightarrow{\nabla u}) = 0 & \text{if } x \in \Omega \\ u(x) = \varphi(x) & \text{if } x \in \partial\Omega, \end{cases}$$

where

$$H(x,p) = I(x)\sqrt{1+|p|^2} + p \cdot 1 - \gamma.$$

To do that, we apply the method described in section 3.1. The reader most probably remembers that this method leads to the maximisation (for $q \in -D_x = \overline{B}(-1, I(x))$) of the function:

$$K_x(q) = q \cdot \begin{pmatrix} A(\Delta x_1, x, u(x), u, q) \\ B(\Delta x_2, x, u(x), u, q) \end{pmatrix} - H^*(x, -q); \tag{4.13}$$

where

$$A(\Delta x_1, x, u(x), u, q) = \begin{cases} -a(\Delta x_1, x, u(x), u) & \text{if } q_1 \leq 0, \\ b(\Delta x_1, x, u(x), u) & \text{if } q_1 > 0, \end{cases}$$

and

$$B(\Delta x_2, x, u(x), u, q) = \begin{cases} -c(\Delta x_2, x, u(x), u) & \text{if } q_2 \leq 0, \\ d(\Delta x_2, x, u(x), u) & \text{if } q_2 > 0. \end{cases}$$

Recall that:

$$\begin{array}{rcl} a(\rho,x,t,u) & = & \frac{t-u(x-(\rho,0))}{\rho}, \\ b(\rho,x,t,u) & = & \frac{t-u(x+(\rho,0))}{\rho}, \\ c(\rho,x,t,u) & = & \frac{t-u(x-(0,\rho))}{\rho}, \\ d(\rho,x,t,u) & = & \frac{t-u(x+(0,\rho))}{\rho}. \end{array}$$

As we have seen in section 3.1, to maximise this function we need to partition the set D into four subsets D_1, D_2, D_3 and D_4 (defined by (3.7)), and to compute independently the *sup* over each subset.

- First, note that, for all subsets D_i, we have to consider functions like:

$$\kappa_x^{\theta_1\theta_2}(q) = q \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \sqrt{I(x)^2 - |q+1|^2} - \gamma$$

$\kappa_x^{\theta_1\theta_2}$ is defined on $B(-1, I(x))$. Using differential calculus, it is easy to find the global maximum of $\kappa_x^{\theta_1\theta_2}$ over $B(-1, I(x))$. We have:

$$\nabla\kappa_x^{\theta_1\theta_2}(q) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \frac{q+1}{\sqrt{I(x)^2 - |q+1|^2}}$$

Thus

$$\begin{array}{rl} \nabla\kappa_x^{\theta_1\theta_2}(q_0) = 0 \iff & q_0 + 1 = \sqrt{I(x)^2 - |q_0+1|^2} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\ \implies & \exists\lambda \geq 0 \quad | \quad q_0 + 1 = \lambda \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \end{array}$$

A short computation shows that:

$$\lambda = \frac{I(x)}{\sqrt{1 + |\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}|^2}}.$$

The maximum is then reached at

$$q_0 = -1 + I(x) \frac{\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}}{\sqrt{1 + |\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}|^2}};$$

obviously $q_0 \in B(-1, I(x))$. Replacing the value q_0 in $\kappa_x^{\theta_1\theta_2}$ we obtain the maximal value of $\kappa_x^{\theta_1\theta_2}$:

$$\kappa_x^{\theta_1\theta_2}(q_0) = I(x)\sqrt{1 + |\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}|^2} - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \cdot 1 - \gamma.$$

- To simplify matters but without loss of generality, we apply a rotation of axis Oz, and assume that $\beta = 0$ and $\alpha > 0$.
- Let us detail the steps for D_2; on this domain,

$$\begin{aligned} K_x(q) &= -q_1 a(\Delta x_1, x, u(x), u) + q_2 d(\Delta x_2, x, u(x), u) - H^*(x, -q), \\ &= q \cdot \begin{pmatrix} -a(\Delta x_1, x, u(x), u) \\ d(\Delta x_2, x, u(x), u) \end{pmatrix} - H^*(x, -q). \end{aligned}$$

According to the previous computation, the maximum over $B(-\mathbf{1}, I(x))$ is reached at

$$q_0 = -\mathbf{1} + I(x) \frac{\begin{pmatrix} -a \\ d \end{pmatrix}}{\sqrt{1 + a^2 + d^2}}.$$

1. Let us first suppose that $I(x) > \alpha$.
   - If $d \leq 0$ then $q_0 \in D_3 \cup D_4$. Geometrically and using some convexity arguments, it is clear that the maximum on D_2 is reached on $[-\alpha - I(x), 0] \times \{0\}$. An easy maximisation in dimension one yields

$$\max_{q \in D_2} K_x(q) = I(x) \sqrt{1 + [\max(a, -\frac{\alpha}{\sqrt{I(x)^2 - \alpha^2}})]^2} + \alpha \max(a, -\frac{\alpha}{\sqrt{I(x)^2 - \alpha^2}}) - \gamma.$$

   - If $d > 0$ then $q_0 \in D_1 \cup D_2$
     - If $q_0 \in D_1$ then the maximum over D_2 is reached on $\{0\} \times [0, \sqrt{I(x)^2 - \alpha^2}]$ and its value is :

$$\max_{q \in D_2} K_x(q) = \sqrt{I(x)^2 - \alpha^2} \sqrt{1 + \alpha^2} - \gamma.$$

     - If $q_0 \in D_2$ then $\max_{D_2} K_x$ is $K_x(q_0)$:

$$\max_{q \in D_2} K_x(q) = I \sqrt{1 + a^2 + d^2} + \alpha\ a - \gamma.$$

   We can synthetise all these cases by:

$$\max_{q \in D_2} K_x(q) = I(x) \sqrt{1 + (\chi^+(a, d^+))^2 + (d^+)^2} + \alpha \chi^+(a, d^+) - \gamma$$

   where

   * the operator $\cdot^+$ is defined by

$$\cdot^+ : \mathbb{R} \longrightarrow \mathbb{R} : d \longmapsto d^+ = \begin{cases} d & \text{if } d \geq 0 \\ 0 & \text{if } d \leq 0, \end{cases}$$

∗ the operator $\cdot^-$ by

$$\cdot^- : \mathbb{R} \longrightarrow \mathbb{R} : d \longmapsto d^- = \begin{cases} -d & \text{if } d \leq 0 \\ 0 & \text{if } d \geq 0, \end{cases}$$

∗ the functions $\chi^+(x,y)$ and $\chi^-(x,y)$ are defined by

$$\chi^+(x,y) = \begin{cases} x & \text{if } x \geq -\alpha\frac{\sqrt{1+y^2}}{\sqrt{I^2-\alpha^2}} \\ -\alpha\frac{\sqrt{1+y^2}}{\sqrt{I^2-\alpha^2}} & \text{otherwise,} \end{cases}$$

and

$$\chi^-(x,y) = \begin{cases} x & \text{if } x \leq -\alpha\frac{\sqrt{1+y^2}}{\sqrt{I^2-\alpha^2}} \\ -\alpha\frac{\sqrt{1+y^2}}{\sqrt{I^2-\alpha^2}} & \text{otherwise,} \end{cases}$$

(see the figures (4.2) and (4.3)).

2. In the case where $I(x) \leq \alpha$, we note that D_1 and D_4 are empty; In this case, the maximum is

$$\max_{q \in D_2} K_x(q) = I(x)\sqrt{1 + a^2 + (d^+)^2} + \alpha a - \gamma.$$

- Using exactly the same method, we can calculate the maximum on the three other subsets D_1, D_3 and D_4. We then collect the results of the maximization on all the subsets, let $\Delta x_1 = \Delta x_2 = \rho$, and obtain the following numerical scheme:

$$S'(\rho, x, u(x), u) = 0, \tag{4.14}$$

where

$$S'(\rho, x, t, u) = \rho\ S(\rho, x, t, u),$$

and S is defined by:

1. If $x \in \Omega^\rho$ and $I(x) > \alpha$ then

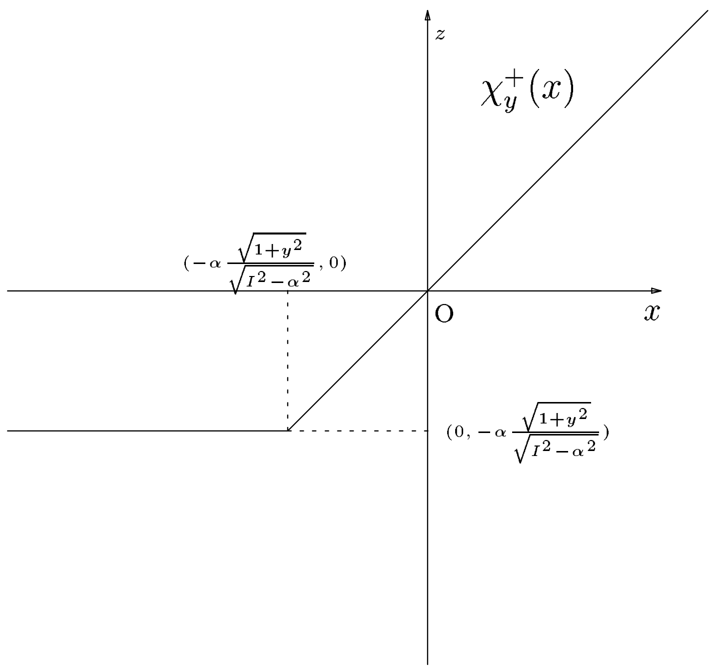
$$S(\rho, x, t, u) = \max(\ K_1(a,c,d), K_2(b,c,d)\ ), \tag{4.15}$$

where

$$\begin{aligned} K_1(a,c,d) &= I(x)\sqrt{1 + (\chi^+(a,M))^2 + M^2} + \alpha\chi^+(a,M) - \gamma, \\ K_2(b,c,d) &= I(x)\sqrt{1 + (\chi^-(-b,M))^2 + M^2} + \alpha\chi^-(-b,M) - \gamma, \\ M &= \max(c^+, d^+), \end{aligned} \tag{4.16}$$

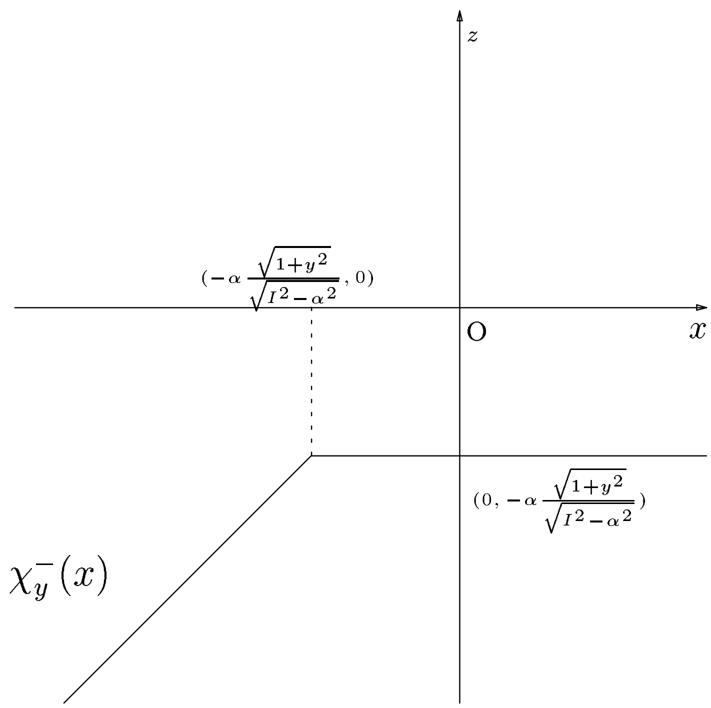
and

$$a = \frac{t - u(x - (\rho,0))}{\rho} \quad b = \frac{t - u(x + (\rho,0))}{\rho} \quad c = \frac{t - u(x - (0,\rho))}{\rho}$$
$$d = \frac{t - u(x + (0,\rho))}{\rho}. \tag{4.17}$$



$$\begin{array}{rccl} \chi_y^+(\cdot): & \mathbb{R} & \rightarrow & \mathbb{R} \\ & x & \mapsto & \chi_y^+(x) = \chi^+(x,y) \end{array}$$

Figure 4.2: Graph of the function $\chi^+(\cdot, y)$.



$$\begin{array}{rccl} \chi_y^-(\cdot): & \mathbb{R} & \rightarrow & \mathbb{R} \\ & x & \mapsto & \chi_y^-(x) = \chi^-(x,y) \end{array}$$

Figure 4.3: Graph of the function $\chi^-(\cdot, y)$.

2. If $x \in \Omega^\rho$ and $I(x) \leq \alpha$ then $S(\rho, x, t, u) = I(x)\sqrt{1 + a^2 + M^2} + \alpha a - \gamma$.
3. If x$\in \flat\Omega^\rho$ then

$$S(\rho, x, t, u) = t - \varphi(x). \tag{4.18}$$

Recall that we have noted $\Omega^\rho = \{x \in \Omega | x - (\rho, 0), x + (\rho, 0), x - (0, \rho), x + (0, \rho) \in \overline{\Omega}\}$ and $\flat\Omega^\rho = \overline{\Omega} - \Omega^\rho$.

4.2.2 Convergence of our approximation scheme

- Verification of the hypotheses of proposition 5:

  1. g is nondecreasing with respect to a, b, c and d:

     (a) if $x \in \Omega^\rho$ and $I(x) > \alpha$:

     $$g(x, a, b, c, d) = \max(K_1(a, c, d), K_2(b, c, d)),$$

     where K_1 and K_2 are described in (4.16). To prove that g is nondecreasing with respect to a, b, c and d, it is sufficient to prove that K_1 and K_2 are.
     $\triangle$ We prove that K_1 is nondecreasing with respect to a, b, c and d.

     i. Let us consider the function g_z.

     $$\begin{array}{rccl} g_z: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & I(x)\sqrt{1 + \theta^2 + z^2} + \alpha\theta - \gamma, \end{array} \tag{4.19}$$

     The graph and the variations of this function are given in figure 4.4.

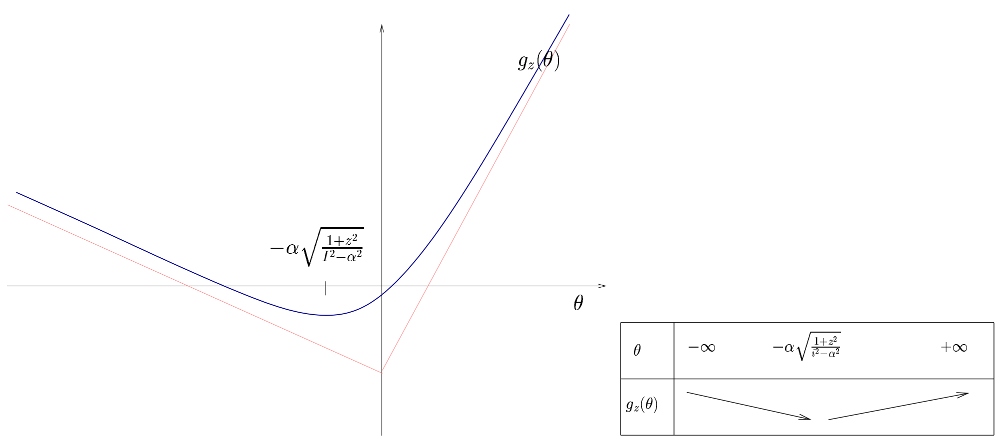


Figure 4.4: Study of g_z

It is easy to see that the graph of the application $\mathcal{K}_z$

$$\begin{array}{rccl} \mathcal{K}_z: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & I(x)\sqrt{1+\chi^+(\theta,z)^2+z^2}+\alpha\chi^+(\theta,z)-\gamma, \end{array}$$

is as shown in figure 4.5 and that $\mathcal{K}_z$ is a nondecreasing function.

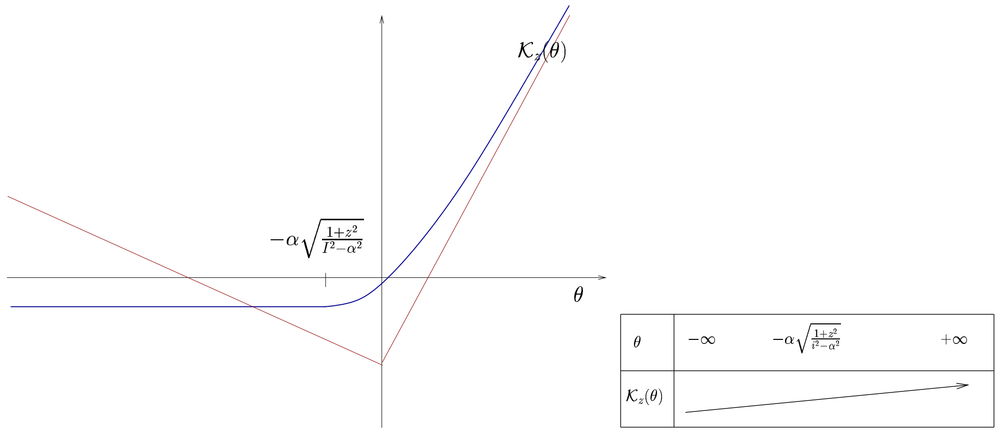


Figure 4.5: Study of $\mathcal{K}_z$

ii. Notice that if $|z_1| \geq |z_2|$ then $g_{z_1} \geq g_{z_2}$ (see figure 4.6). Thus, if $|z_1| \geq |z_2|$ then $\mathcal{K}_{z_1} \geq \mathcal{K}_{z_2}$ (see figure 4.7).

Following the notations of (4.16) M is positive and nondecreasing with respect to c and d. So by composition of function, K_1 is also nondecreasing with respect to c and d.

□

To prove that K_2 is nondecreasing, the reader will verify that the function $\mathcal{K}_z^-$

$$\begin{array}{rccl} \mathcal{K}_z^-: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & I(x)\sqrt{1+\chi^-(\theta,z)^2+z^2}+\alpha\chi^-(\theta,z)-\gamma, \end{array}$$

is nonincreasing and will conclude that K_2 is nondecreasing with respect to b.

(b) if $x \in \Omega^\rho$ and $I(x) \leq \alpha$, the function g_z defined by (4.19) is nondecreasing and the conclusion follows easily.

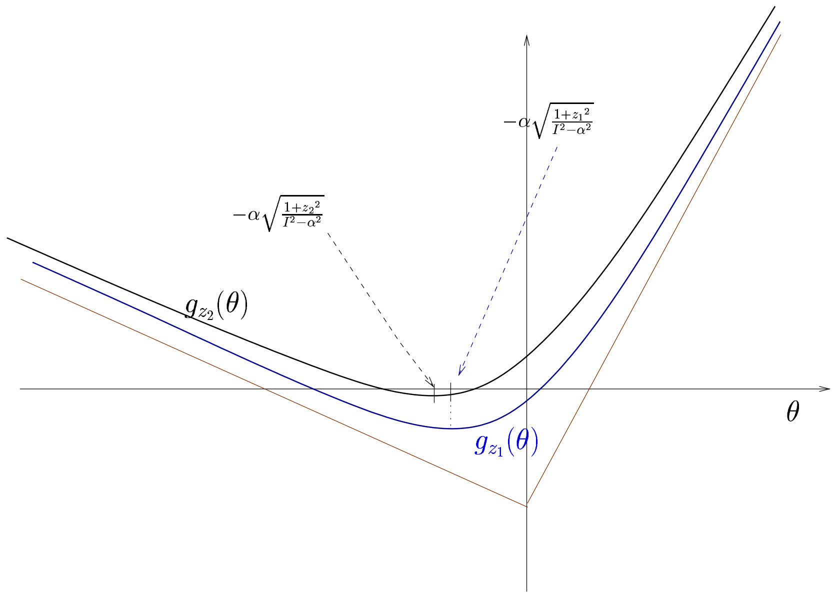


Figure 4.6: comparison of g_{z_1} and g_{z_2} when $|z_1| \geq |z_2|$

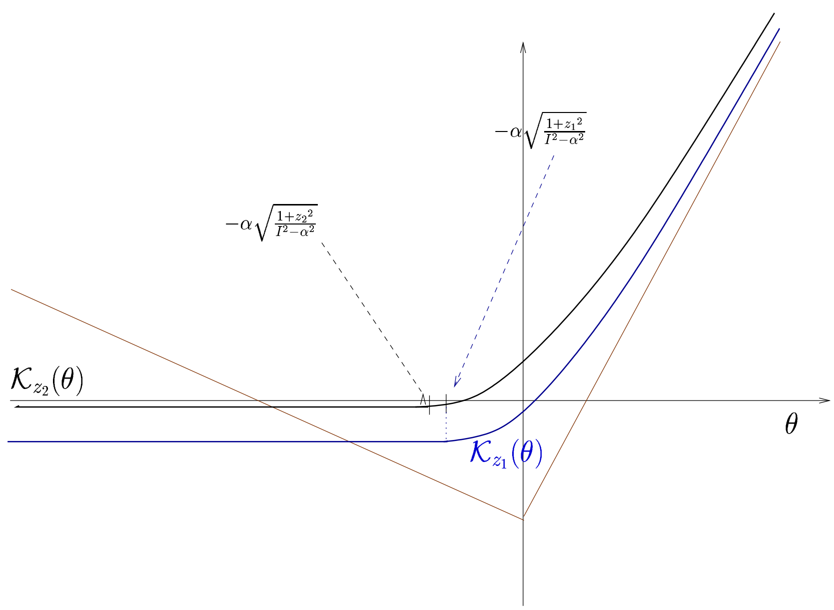


Figure 4.7: comparison of $\mathcal{K}_{z_1}$ and $\mathcal{K}_{z_2}$ when $|z_1| \geq |z_2|$

2. Existence of a bounded function u_0 such $\forall x \in \overline{\Omega},\ S(\rho, x, u_0(x), u_0) \leq 0$:
We define

$$\begin{cases} u_0(x_1, x_2) &= -\frac{\alpha}{\gamma}(x_1 - m_1) + \min_{z \in \flat\Omega^\rho} \varphi(z) & \forall (x_1, x_2) \in \Omega^\rho; \\ u_0(x_1, x_2) &= \varphi(x_1, x_2) & \forall (x_1, x_2) \in \flat\Omega^\rho. \end{cases}$$

where $m_1 = \min\{x_1 \mid (x_1, x_2) \in \overline{\Omega}\}$.
We show that $S(\rho, x, u_0(x), u_0) \leq 0$ (see figure 4.8):

(a) if $x \in \Omega^{2\rho}$ and $I(x) > \alpha$: (note that x is not in Ω^ρ but in $\Omega^{2\rho}$)

$$\begin{aligned} S(\rho, x, u_0(x), u_0) &= g(x,\ -\frac{\alpha}{\gamma},\ \frac{\alpha}{\gamma},\ 0,\ 0) \\ &= \max(\ K_1(-\frac{\alpha}{\gamma}, 0, 0),\ K_2(\frac{\alpha}{\gamma}, 0, 0)\ ) \end{aligned} \tag{4.20}$$

- $K_1(-\frac{\alpha}{\gamma}, 0, 0) \leq 0$:
  In effect, since $-\frac{\alpha}{\gamma} = -\frac{\alpha}{\sqrt{1-\alpha^2}} \geq -\frac{\alpha}{\sqrt{I(x)^2 - \alpha^2}}$ we have $\chi^+(-\frac{\alpha}{\gamma}, 0) = -\frac{\alpha}{\gamma}$, and therefore

$$K_1(-\frac{\alpha}{\gamma}, 0, 0) = I(x)\sqrt{1 + \frac{\alpha^2}{\gamma^2}} - \frac{\alpha^2}{\gamma} - \gamma \leq 0.$$

- $K_2(\frac{\alpha}{\gamma}, 0, 0) \leq 0$:
  For the same reasons,

$$\chi^-(-\frac{\alpha}{\gamma}, 0) = -\frac{\alpha}{\sqrt{I(x)^2 - \alpha^2}}.$$

  Hence

$$K_2(\frac{\alpha}{\gamma}, 0, 0) = I(x)\sqrt{1 + \left(\frac{-\alpha}{\sqrt{I(x)^2 - \alpha^2}}\right)^2} + \alpha\left(\frac{-\alpha}{\sqrt{I(x)^2 - \alpha^2}}\right) - \gamma \leq 0.$$

  So

$$K_2(\frac{\alpha}{\gamma}, 0, 0) \leq 0.$$

(b) if $x \in \Omega^{2\rho}$ and $I(x) \leq \alpha$:

$$S(\rho, x, u_0(x), u_0) = I(x)\sqrt{1 + \left(\frac{-\alpha}{\gamma}\right)^2} - \frac{\alpha^2}{\gamma} - \gamma \leq 0$$

(c) if $x \in \flat\Omega^\rho$: $S(\rho, x, u_0(x), u_0) = 0$

(d) if $x \in \Omega^\rho - \Omega^{2\rho}$:
It is a little more difficult but it is the same idea. The reader can convince himself by looking at figure 4.8 and noticing that
- We always have $a \leq \frac{-\alpha}{\gamma}$,$b \leq \frac{-\alpha}{\gamma}$, $c \leq 0$ and $d \leq 0$;
- since $c \leq 0$ and $d \leq 0$ then $M = 0$ and $K_i(.,c,d) = K_i(.,0,0)$;
- K_1 and K_2 being nondecreasing, we have:

$$K_1(a,0,0) \leq 0 \quad \text{and} \quad K_2(b,0,0) \leq 0.$$

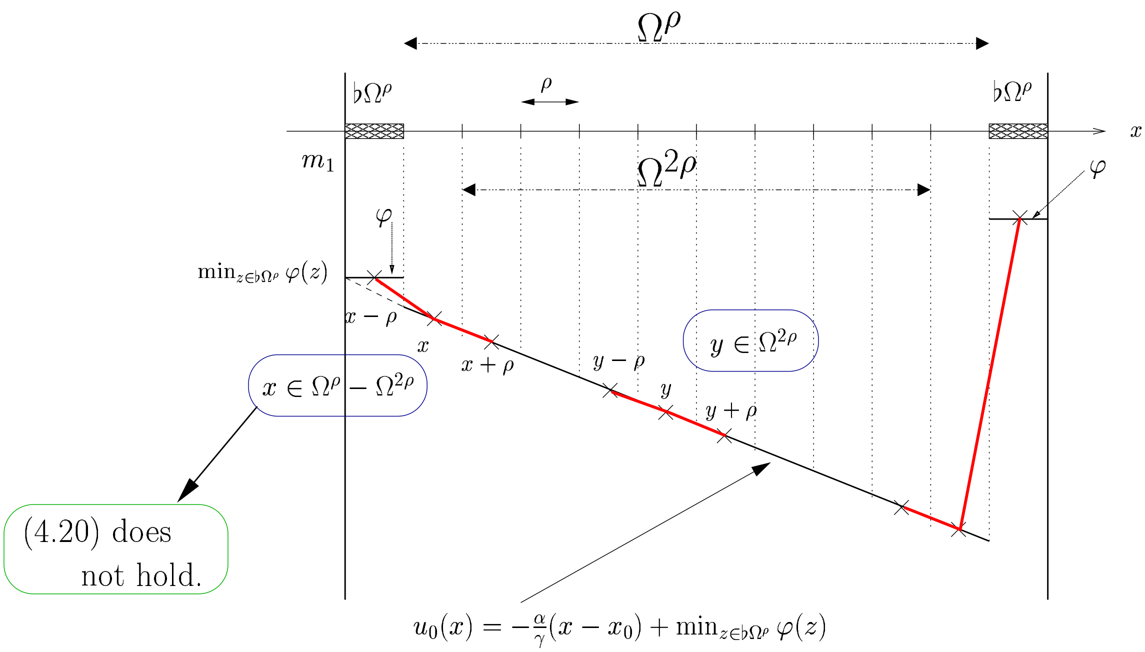


Figure 4.8: study of u_0 for $x \in \Omega^\rho - \Omega^{2\rho}$

3. For all ρ, x and u fixed, $\lim_{t\to+\infty} S(\rho,x,t,u) \geq 0$:
since when $t \to +\infty$,

$$a(\rho,x,t,u) \longrightarrow +\infty, \quad b(\rho,x,t,u) \longrightarrow +\infty,$$

$$c(\rho,x,t,u) \longrightarrow +\infty \quad \text{and} \quad d(\rho,x,t,u) \longrightarrow +\infty,$$

it is easy to see that $g(x,a,b,c,d) \longrightarrow +\infty$.

4. $g(x,a,b,c,d) \leq 0 \Longrightarrow a$ is upper-bounded independently of x:
Note that for all M,

$$I(x)\sqrt{1+\left(\frac{\gamma}{\alpha}\right)^2+M^2}+\alpha\frac{\gamma}{\alpha}-\gamma \geq \frac{I(x)}{\alpha} > 0$$

and that $\frac{\alpha}{\gamma} \geq 0$ (so that $\chi^+(\frac{\gamma}{\alpha}, M) = \frac{\gamma}{\alpha}$). Thus, letting $a = \frac{\gamma}{\alpha}$ we have for all c and d, $K_1(a,c,d) > 0$ and therefore

$\forall x \in \Omega^\rho$ such that $I(x) > \alpha$, $\qquad g(x,a,b,c,d) = \max(K_1(a,c,d), K_2(b,c,d)) > 0.$

The case where $I(x) \leq \alpha$ is now obvious. Conclusion, by nondecreasing of g in respect to a, we have:

$$g(x,a,b,c,d) \leq 0 \quad \Longrightarrow \quad a < \frac{\gamma}{\alpha}.$$

5. By composition, it is clear that g is continuous in respect to a, b, c and d.
Finally, remark that

$$\begin{aligned}\max(K_z(\theta), K_z^-(\theta)) &= \max(I(x)\sqrt{1+\chi^+(\theta,z)^2+z^2}+\alpha\chi^+(\theta,z)-\gamma,\\ &\qquad I(x)\sqrt{1+\chi^-(\theta,z)^2+z^2}+\alpha\chi^-(\theta,z)-\gamma)\\ &= I(x)\sqrt{1+\theta^2+z^2}+\alpha\theta-\gamma = g_z(\theta);\end{aligned}$$

as illustrated on the figure (4.9). Thus, $\forall x \in \Omega^\rho$ such as $I(x) > \alpha$, and $\forall \phi \in$

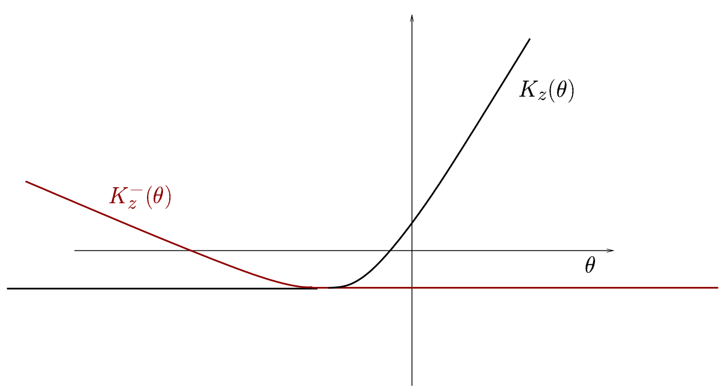


Figure 4.9: $\max(K_z(\theta), K_z^-(\theta)) = g_z(\theta)$

$C_b^\infty(\overline{\Omega})$:

$$g(x,\ \partial_x\phi(x),\ -\partial_x\phi(x),\ \partial_y\phi(x),\ -\partial_y\phi(x))$$

$$
\begin{aligned}
&= I(x)\sqrt{1+\partial_x\phi(x)^2+\partial_y\phi(x)^2}+\alpha\partial_x\phi(x)-\gamma \\
&= H(x,\nabla\phi(x)).
\end{aligned}
$$

For x in Ω^ρ such as $I(x) \leq \alpha$, we have clearly the same result.

- In section 4.1, we have shown that the strong uniqueness theorem applies to the general equation iff for all x in $\overline{\Omega}$, $\quad |\mathbf{l}| < I(x) < 1$.
- Conclusion of this section:
  - For all image I ($I(x) \in [0,1]$), the hypotheses of the proposition 5 are verified for our scheme. Thus for any images I, the scheme is consistent with the equation (4.4) and has a solution uro.
  - The hypotheses of the strong uniqueness theorem are true iff $1 > I > |\mathbf{l}|$.

  Consequently, as soon as $\forall x \in \overline{\Omega}$, $1 > I(x) > |\mathbf{l}|$ then by the theorem 8, the solutions u^ρ of our scheme (4.14) converge toward the viscosity solution of equation (4.4).

4.3 Algorithm for computing the solution

4.3.1 Algorithm for the general equation

We use again the notations of section 3.3 and apply the method we proposed there.
We assume that the image I is known on the discrete grid Q.
Recall that the algorithm consists of the following computation of the sequence of values U_{ij}^n, $n \geq 0$:

Algorithm 2 *1. Initialisation ($n=0$): $U_{ij}^0 = u_0(x_{ij})$.*

2. Choice of a pixel $x_{ij} \in Q^\rho$ and modification (step $n+1$) of U_{ij}^n:
We choose $U^{n+1} = \sup\{V = (V_{k,l})_{x_{kl}\in Q}$ such that $\forall (k,l) \neq (i,j), V_{kl} = U_{kl}^n$ and $g(\rho, x_{ij}, V_{ij}, V) = 0\}$.

3. Choose the next pixel $x_{ij} \in Q^\rho$ in such a way that all pixels of Q^ρ are regulary visited and go back to 2.

We now go into more detail and deal fully with step 2 of algorithm 2 for a point x_{ij} such that $I(x_{ij}) > \alpha$. Let us note

$$K_1(t) = I(x)\sqrt{1+(\chi^+(D_x^- U_{ij}(t), M(t)))^2 + M(t)^2} + \alpha\chi^+(D_x^- U_{ij}(t), M(t)) - \gamma,$$

$$K_2(t) = I(x)\sqrt{1+(\chi^-(D_x^+ U_{ij}(t), M(t)))^2 + M(t)^2} + \alpha\chi^-(D_x^+ U_{ij}(t), M(t)) - \gamma$$

and

$$K(t) = \max(K_1(t), K_2(t)),$$

where

$$D_x^- U_{ij}(t) = \frac{t - U_{i-1j}}{\Delta x_1} \quad D_x^+ U_{ij}(t) = \frac{U_{i+1j} - t}{\Delta x_1}$$
$$D_y^- U_{ij}(t) = \frac{t - U_{ij-1}}{\Delta x_2} \quad D_y^+ U_{ij}(t) = \frac{U_{ij+1} - t}{\Delta x_2};$$

$$M(t) = \max((D_y^- U_{ij}(t))^+, (D_y^+ U_{ij}(t))^-).$$

We note $U_{minj} = \min(U_{i,j+1}, U_{i,j-1})$.

In this case ($I(x_{ij}) > \alpha$), step 2 of algorithm 2 consists of computing the value of t such that $K(t) = 0$, that is to say such that:

$$(\underbrace{K_1(t) = 0 \quad \text{and} \quad K_2(t) \leq 0}_{\text{Case 1}}) \quad \text{or} \quad (\underbrace{K_2(t) = 0 \quad \text{and} \quad K_1(t) \leq 0}_{\text{Case 2}}).$$

Therefore, t is either a root of K_1, or a root of K_2. We can be even more precise: in effect, K_1 and K_2 being nondecreasing functions, they each have at most one root. Moreover they always have one. Thus, the root of K is $t = \min\{r_1, r_2\}$ where r_1 is the unique root of K_1 and r_2 is the unique root of K_2.
$\triangle$ The proof that the functions K_1 and K_2 are nondecreasing is clear using the result of step 1 of the section 4.2.2. For example, it is easy to prove that for all t sufficiently small, $K_1(t)$ is negative and that $\lim_{t \to +\infty} K_1(t) = +\infty$. By continuity, K_1 always has one root. $\square$

Let us determine r_1 and r_2.
We detail the procedure in the case of K_1:

1. Case $t \geq U_{minj}$:
   - If t is such that
   $$D_x^- U_{ij}(t) \geq -\alpha \frac{\sqrt{1 + M(t)^2}}{\sqrt{I^2 - \alpha^2}} \tag{4.21}$$
   then t is a root of
   $$I(x)\sqrt{1 + D_x^- U_{ij}(t)^2 + M(t)^2} + \alpha D_x^- U_{ij}(t) - \gamma \; = \; 0. \tag{4.22}$$
   - If t is such that
   $$D_x^- U_{ij}(t) \leq -\alpha \frac{\sqrt{1 + M(t)^2}}{\sqrt{I^2 - \alpha^2}} \tag{4.23}$$
   then t is a root of
   $$I(x)\sqrt{1 + \alpha^2 \frac{1 + M(t)^2}{I^2 - \alpha^2} + M(t)^2} - \alpha^2 \sqrt{\frac{1 + M(t)^2}{I^2 - \alpha^2}} - \gamma \; = \; 0. \tag{4.24}$$

2. Case $t \leq U_{minj}$. $(M(t) = 0\ !)$
   In that case, t is such that $D_x^- U_{ij}(t) \geq -\frac{\alpha}{\sqrt{I^2-\alpha^2}}$ and therefore it is the root of

$$I(x)\sqrt{1 + D_x^- U_{ij}(t)^2} + \alpha D_x^- U_{ij}(t) - \gamma \;=\; 0. \tag{4.25}$$

In short we must

1. compute the root $\hat{t}$ of (4.22).
2. if $\hat{t}$ exists and verifies (4.21) then $\hat{t}$ is the root of K_1.
3. otherwise we must
   (a) compute the root $\check{t}$ of (4.24).
   (b) if $\check{t}$ exists and verifies (4.23) then $\check{t}$ is the root of K_1.
   (c) otherwise the root of K_1 is the root of (4.25).

Remark: This algorithm has been coded in C++. The code is given in appendix A.

4.3.2 Convergence of the algorithm:

In the section 4.2.2, we have proved that for any image I ($I(x) \in [0,1]$), the hypotheses of the proposition 5 are verified for our scheme. Recall that therefore our scheme always is consistent with the equation (4.4) and always has a solution. Moreover the theorem 9 implies that for all image I, our algorithm converge toward a solution of our scheme. Finally, recall also that if $\forall x \in \overline{\Omega}$, $|\mathbf{l}| < I(x) < 1$ then our scheme has at most one solution u^ρ and that u^ρ converges toward the unique viscosity solution of equation (4.4) when $\rho \to 0$.

4.4 Experimental results

We have tested our algorithm with synthetic images generated by shapes with several levels of regularity e.g. C^∞ (a sinusoid, see figures 4.10 and 4.11), or C^0 (a pyramid, see figures 4.12 and 4.13), to demonstrate the ability of our method to deal with smooth and nonsmooth objects. We also have tested it with real images, an example is shown in figure 4.18.

In all results, the parameters are n, the number of iterations, ϵ_1, ϵ_2 and ϵ_∞ the relative errors of the computed surface measured according to the L_1, L_2 and L_∞ norms, respectively, θ the angle of the direction of illumination with the z-axis.

In all synthetic cases we show the original object, the input image and the reconstructed surface. We then demonstrate the stability of our method with respect to two types of errors. The first type is image intensity errors due to noise. Uniformly distributed noise has

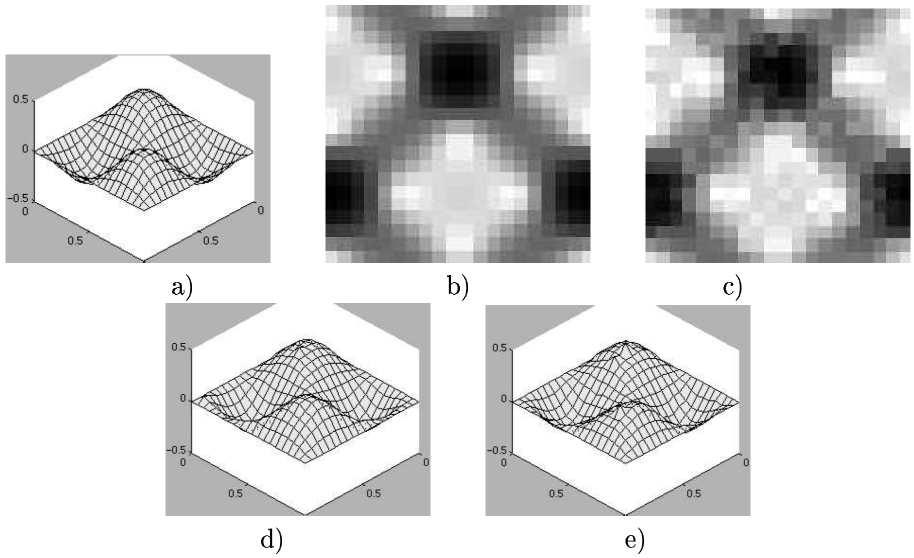


Figure 4.10: Results for a synthetic image generated by a sinusoidal surface sampled on a grid of size 20×20 with $\theta = 28°$: a) original surface, b) original image, c) noisy image; d) surface reconstructed from b): $n = 40$, $\epsilon_1 = 10.0\%$, $\epsilon_2 = 9.9\%$ $\epsilon_\infty = 15.2\%$; e) surface reconstructed from c): $n = 62$, $\epsilon_1 = 12.8\%$, $\epsilon_2 = 10.5\%$ $\epsilon_\infty = 15.7\%$

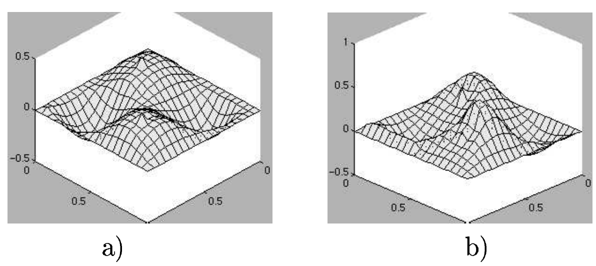
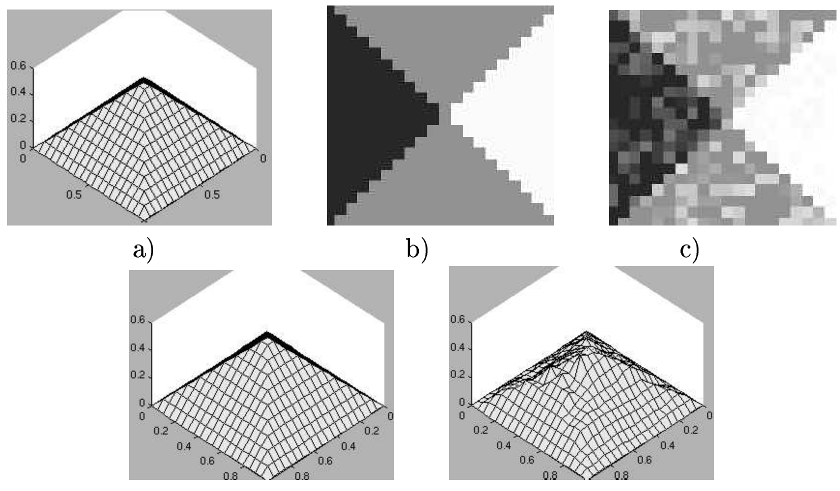


Figure 4.11: Sinusoidal surface of figure 4.10 reconstructed with an error on the parameter $\mathbf{L}$: a) $\theta_p = 33°$, $\epsilon_\theta = 5°$, $n = 37$, $\epsilon_1 = 11.4\%$, $\epsilon_2 = 7.6\%$ $\epsilon_\infty = 17.9\%$; b) $\theta_p = 18°$, $\epsilon_\theta = 10°$, $n = 45$, $\epsilon_1 = 18.3\%$, $\epsilon_2 = 13.8\%$, $\epsilon_\infty = 41.9\%$.

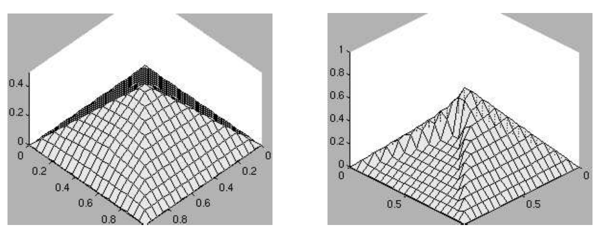


a) b) c)

d) e)

Figure 4.12: Results for a synthetic image generated by a pyramidal surface sampled on a grid of size 20×20 with $\theta = 36°$: a) original surface, b) original image, c) noisy image; d) surface reconstructed from b): $n = 89$, $\epsilon_1 = 0.4\%$, $\epsilon_2 = 0.4\%$ $\epsilon_\infty = 0.8\%$; e) surface reconstructed from c): $n = 91$, $\epsilon_1 = 23.7\%$, $\epsilon_2 = 23.2\%$ $\epsilon_\infty = 26.3\%$.

been added to some pixels of the input images and the corresponding reconstructed surfaces are shown (figures 4.10 and 4.12). The second type of error is due to an incorrect estimation of the direction of illumination $\mathbf{L}$ (figures 4.11 and 4.13). We start with a smooth sinusoidal object, see figure 4.10. We introduce an error of 5° on the parameter $\mathbf{L}$, see figure 4.11. We note θ_p the angle used for computation; $\epsilon_\theta = |\theta - \theta_p|$.



a) b)

Figure 4.13: Pyramidal surface of figure 4.12 reconstructed with an error on the parameter $\mathbf{L}$: a) $\theta_p = 41°$, $\epsilon_\theta = 5°$, $n = 80$, $\epsilon_1 = 16.5\%$, $\epsilon_2 = 15.2\%$ $\epsilon_\infty = 19.8\%$; b) $\theta_p = 26°$, $\epsilon_\theta = 10°$, $\epsilon_1 = 25.1\%$, $\epsilon_2 = 14.0\%$ $\epsilon_\infty = 28.3\%$.

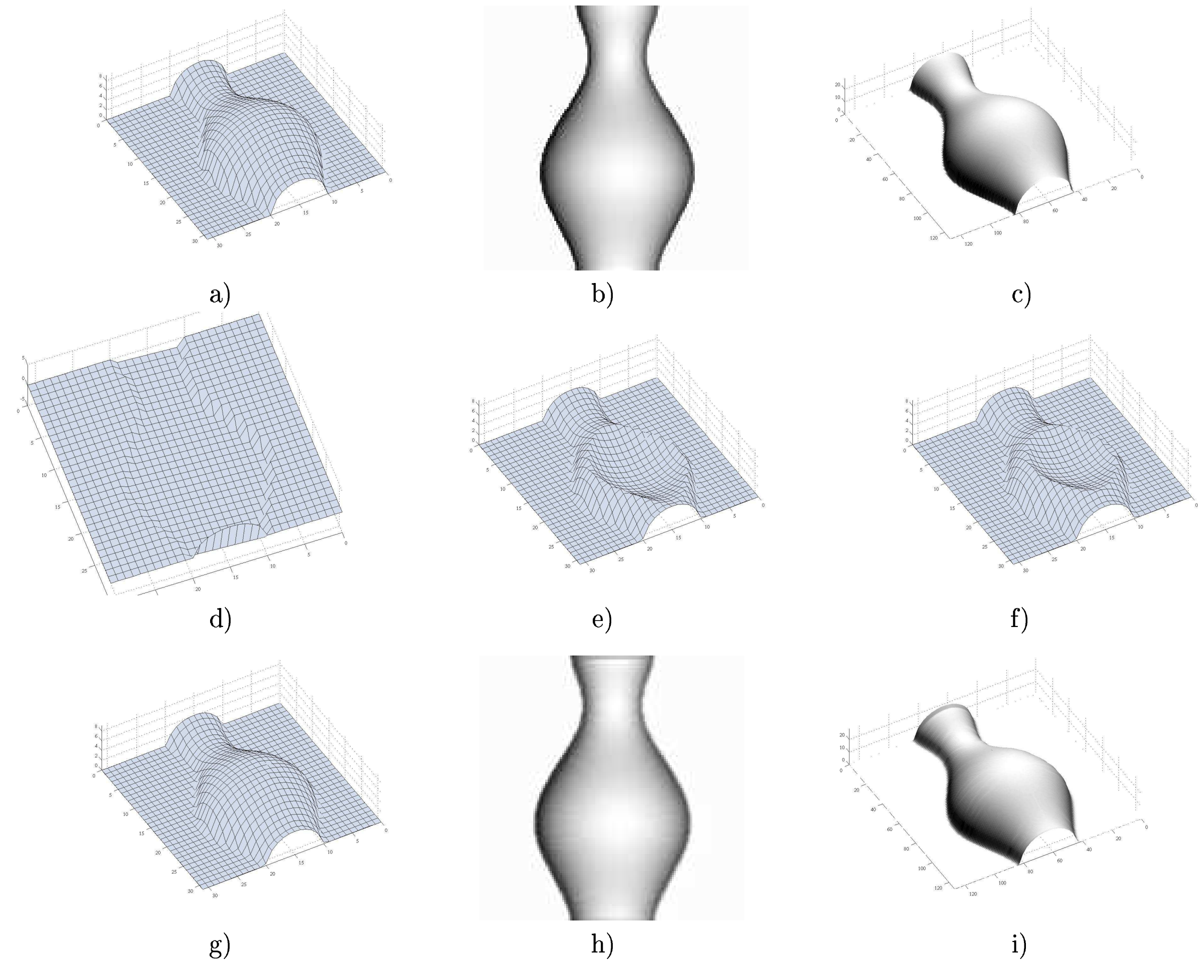
As seen from these figures, our algorithm seems to be quite robust, not only to intensity noise (see figures 4.10.e and 4.12.e), as in [29], but also to inaccuracies in the estimation of the direction of the light source $\mathbf{L}$ (see figures 4.11 and 4.13). The pyramid example shows the remarkable ability of the numerical scheme to deal with functions which are only continuous. This example also shows the convergence of our algorithm with discontinuous images: through the recent works of Ostrov [27, 24], we hope to extend our theory to the case of discontinuous images.

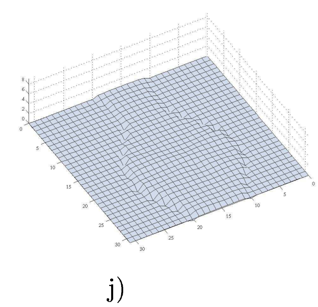
To compare our algorithm to the others, we have tested it with the classical example of the vase represented in party a) and b) of figures 4.14 and 4.15. Figure 4.14 shows the results obtained with $\theta \cong 5.7°$ and figure 4.15 shows the results obtained with $\theta \cong 53°$. In both experiments we have hand-segmented the image and enforced the boundary conditions suggested by the theorical analysis. Note that in oder to obtain satisfactory results it is necessary to take into account the cast shadows. To compare this results the reader can see [30].

Remark: The synthetic surface of the vase and other surfaces are available by anonymous ftp under the pub/tech_paper/survey directory at eustis.cs.ucf.edu (132.170.108.42).

Figure 4.16 illustrates the problems linked to the gamma correction, un probleme pas negligeable en pratique. We have tested our algorithm with corrupted image I^γ with the images of the vase. For $\gamma = 2$, results are somewhat disappointing, see 4.16.

The real image shown in figure 4.18.a and 4.18.b is a photograph of a Halloween mask taken with a standard camera with 70mm focal length. The light source was far from being pointlike and at infinity and ambiant lighting due to reflections on the walls was present. The reflectance of the mask was not quite Lambertian (some highlights were visible). Unlike the case of the synthetic examples where the critical points x such that $I(x) = 1$ were included in the boundary conditions (i.e. their distances were supposed to be known), these distances must in this case be computed by the algorithm, making the problem ill-posed. Despite this difficulty, the results shown in figure 4.18.c-f are of good quality.

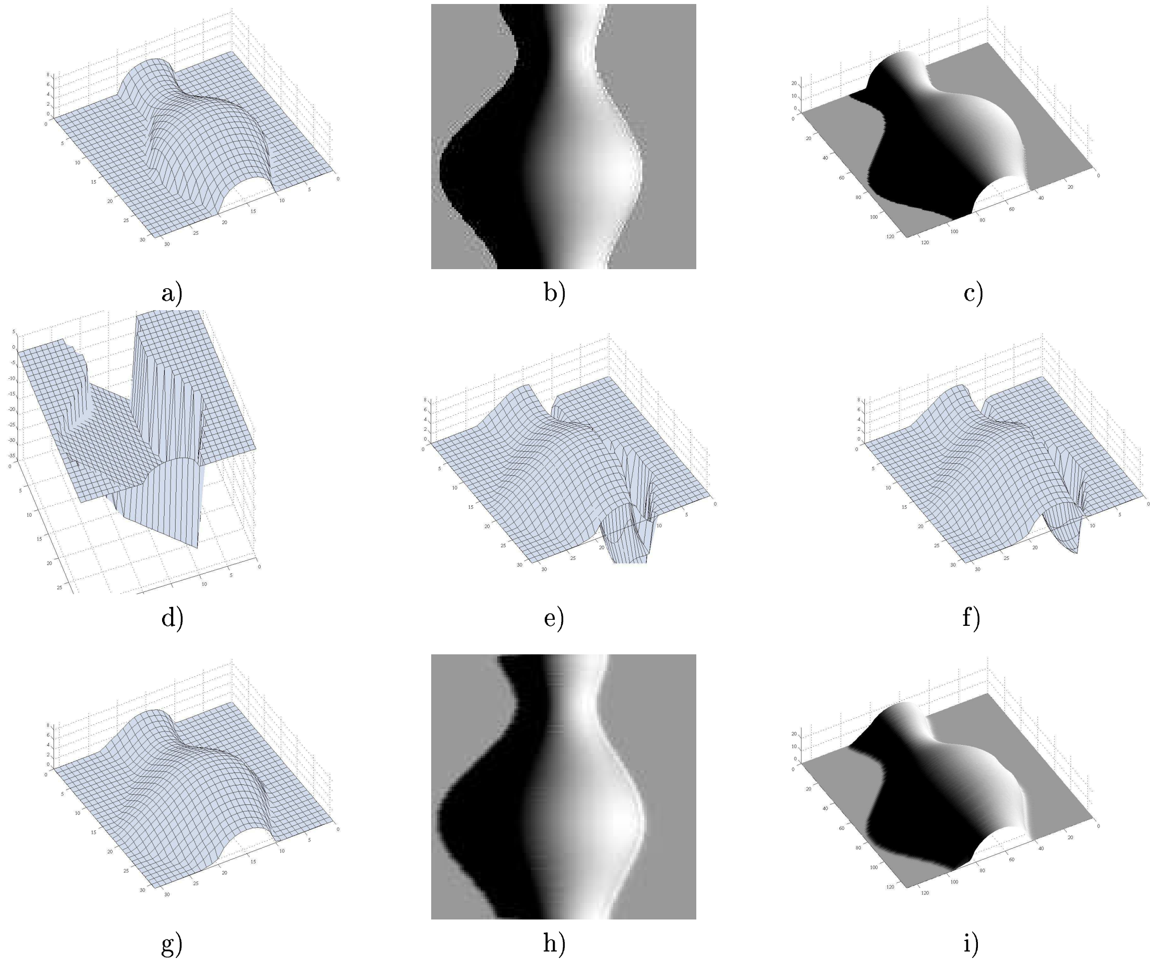


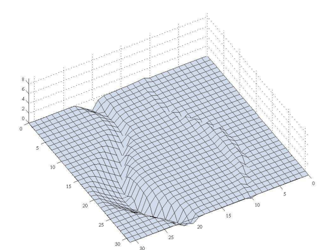


a) Original image of size 128x128, b) synthetic image generated from the original surface, c) Original shaded surface, d) Initial surface, e) and f) Surface U^n at the n^{th} iteration, $n = 500$ and $n = 1000$, respectively. g) Final result, h) Synthetic image generated from the surface g), i) Reconstructed surface, textured with image h), j) Error image.

$$n < 2000, \qquad \epsilon_1 \cong 10.4\%,\ \epsilon_2 \cong 1.0\%\ \epsilon_\infty \cong 9.8\%;$$
$$\mathbf{L} = (\alpha, \beta, \gamma) = (0.1,\ 0.0,\ 0.995) \qquad \Rightarrow \theta = 5.7°.$$

Figure 4.14: Experimental results with a synthetic image representing a vase, $\theta = 5.7°$

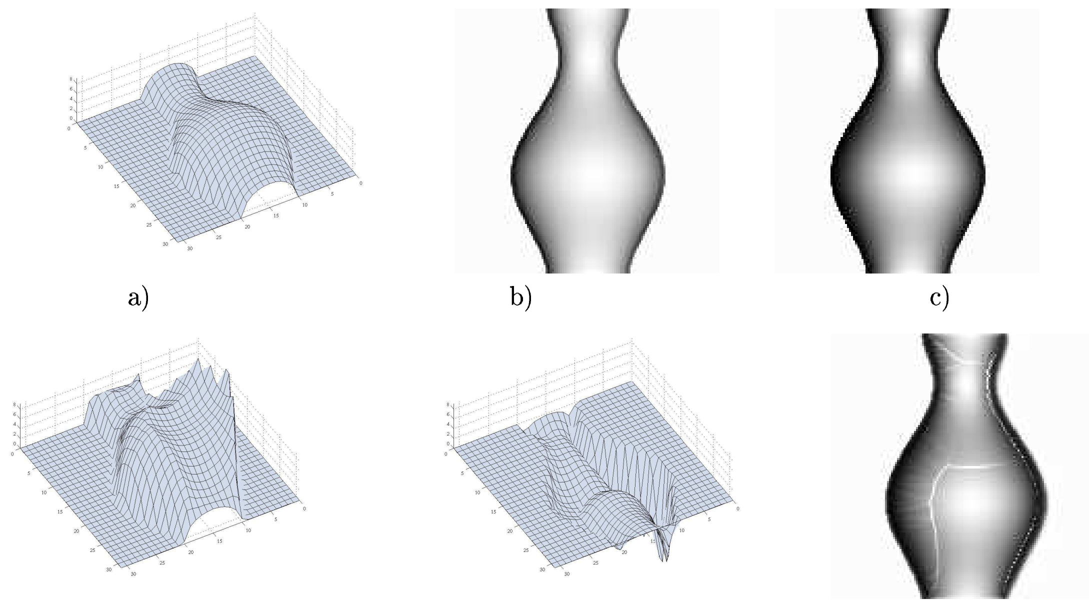




a) Original image of size 128x128, b) Synthetic image generated from the original surface, c) Original shaded surface, d) Initial surface, e) and f) Surface U^n at the n^{th} iteration, $n = 500$ and $n = 1000$, respectively. g) Final result, h) Synthetic image generated from the surface g), i) Reconstructed surface, textured with image h), j) Error image.

$$n < 2500, \qquad \epsilon_1 \cong 16.1\%,\ \epsilon_2 \cong 5.2\%\ \epsilon_\infty \cong 49.2\%;$$
$$\mathbf{L} = (\alpha, \beta, \gamma) = (0.8,\ 0.0,\ 0.6) \qquad \Rightarrow \theta = 53^\circ.$$

Figure 4.15: Experimental results with a synthetic image of a vase, $\theta = 53^\circ$.



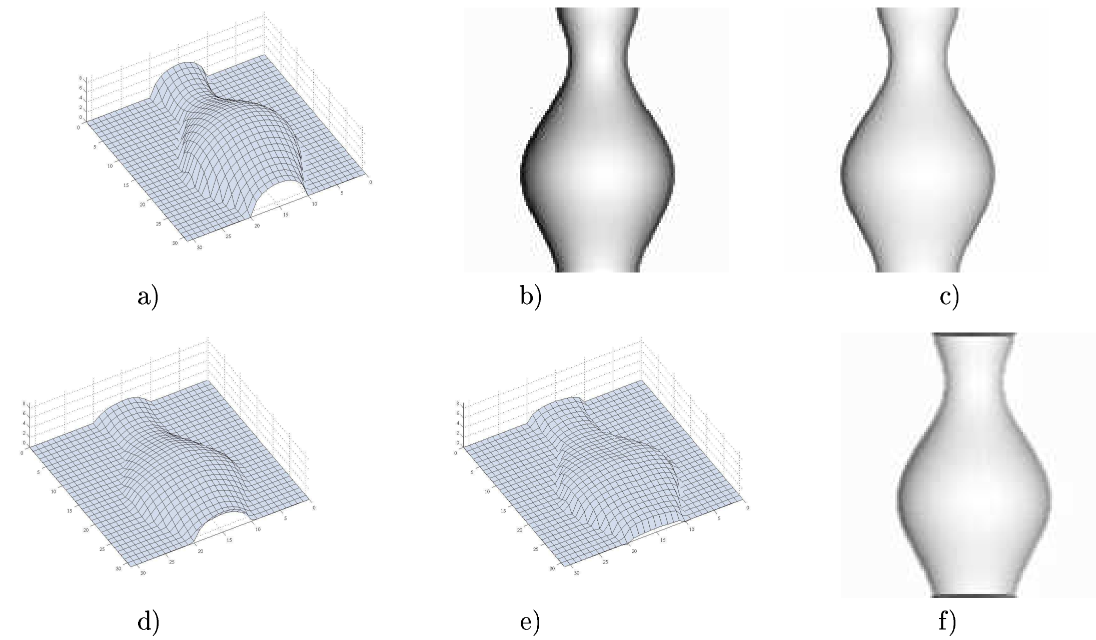
a) b) c)

d) e) f)

a) Original image of size 128x128, b) Synthetic image generated from the original surface, c) Corrupted photo, d) Final result, e) Error , f) Synthetic image generated from the surface d).

$$n < 3500,\ \epsilon_1 \cong 79.4\%,\ \epsilon_2 \cong 79.0\%\ \epsilon_\infty \cong 202.4\%;$$
$$\mathbf{L} = (\alpha, \beta, \gamma) = (0.1,\ 0.0,\ 0.995) \qquad \Rightarrow \theta = 15.7°.$$
$$\gamma = 2$$

Figure 4.16: Results with a synthetic image corrupted by a gamma correction with $\gamma = 2$, i.e. instead of using the real image I we used I^2, as if the gamma distortion had not been compensated for.



a) Original image of size 128x128, b) Synthetic image generated from the original surface, c) Corrupted photo, d) Final result, e) Error , f) Synthetic image generated from the surface d).

$$n < 2500,\ \epsilon_1 \cong 50.0\%,\ \epsilon_2 \cong 24.9\%\ \epsilon_\infty \cong 50.3\%;$$
$$\mathbf{L} = (\alpha, \beta, \gamma) = (0.1,\ 0.0,\ 0.995) \qquad \Rightarrow \theta = 15.7^\circ.$$
$$\gamma = 0.5$$

Figure 4.17: Results with a synthetic image corrupted by a gamma correction with $\gamma = 0.5$, i.e. instead of using the real image I we used $\sqrt{I}$.

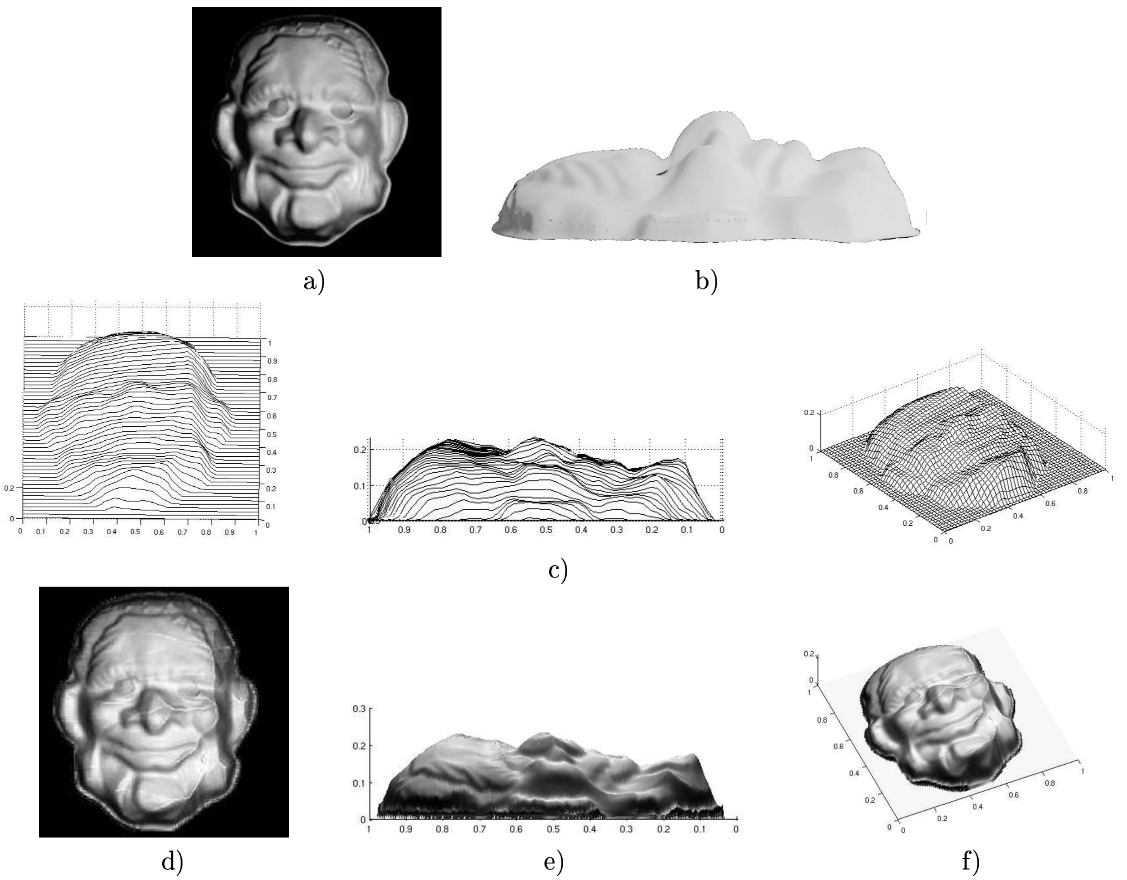


Figure 4.18: Experimental results with a real image: a) Original image of size 200x200, b) Lateral view (image) of the real object, c) Three views of the surface reconstructed from a), d) Synthetic image generated from the surface c), e) and f) reconstructed surface, textured with image d).

Chapter 5

Conclusion

We have proposed a general method allows us to design numerical schemes and algorithms for Hamilton-Jacobi-Bellman equations with null interest rate. We have also proved their convergence towards viscosity solutions. We have applied these results to the simplest version of the shape from shading problem in the case of a Lambertian object illuminated by a point source at infinity and imaged by an orthographic camera. We have spelled out hypotheses for the existence and uniqueness of a (continuous) viscosity solution of the shape from shading problem and proved the convergence of our new numerical scheme in this special case.

We are currently extending our analysis and algorithms to more general cases.

As a side-effect, we also hope to have made the notion of viscosity solutions more accessible and to have convinced the reader of the usability of these tools.

Appendix A

Algorithm (coded in C++)

In this appendix we report our C++ code corresponding to the algorithm 2 described in the section 4.3. We do not show all the details, only the building blocks that will prevent the reader from redoing our computations are given here.

We suppose in the sequel that given functions or procedures are methods included in a class which contains the following variables:

- "U": the current surface,
- "alpha": the first coordinate of the illumination vector $\mathbf{L}$,
  recall that we suppose $\mathbf{L} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ with $\beta = 0$,
- "I": the given image (the photo),
- and other variables used in the methods "rootOfK1", "rootOfK2" and "firstPossibility": "MA","MB", "MC", "MD", "sa","sa2","sb",etc.
  These variables need to be initialised by the constructor calling the "initialiseAndComputeAllVar" method.

Remark that notations are the same as those introduced in section 4.3.

One iteration:

```
void callOneIter(){
    double Umin;
    MAPMASQUE(masque,i,j){
        Umin=min(U(i,j-1),U(i,j+1));
        if (I(i,j)>alpha)     // if I(i,j) > alpha,
                              //  U is the minimum of the root of K1()
```

```
                                //  and the root of K2().
                {
                   U(i,j)=min(rootOfK1(i,j,Umin),rootOfK2(i,j,Umin));
                }
            else                //  if I(i,j) <= alpha,
                                //      if there exists a root >= Umin,
                                //      then this root is given by "firstPossibility(.)",
                                //      else the root is given by "U(i-1,j)+dxl1(i,j)".
                {
                   if ( Umin  >  (U(i,j)=firstPossibility(i,j,Umin)))
                       {   U(i,j)=U(i-1,j)+dxl1(i,j); }
                }
        };
    };
```

Methods giving the roots of K1(t), K2(t):

```
    double rootOfK1(int i,int j,double Umin){

        double res;
        double b,c,delta;
        double Uim1=U(i-1,j);

        res=Uim1+dxl1(i,j);
        if (res > Umin){
            res=Umin+ME(i,j);
            if ( res > (Uim1+P(i,j)) ){
                b=MA(i,j)*Uim1+MB(i,j)*Umin-sa;
                c=MD(i,j)+MA(i,j)*Uim1*Uim1+MB(i,j)*Umin*Umin-sa2*Uim1;
                delta=b*b-MC(i,j)*c;if (delta <0) {delta=0;};

                res=(b+sqrt(delta))/MC(i,j);
                if ( !(
                       (res>=Umin)&
                       (res<=(Uim1+sc))&
                       (
                        ((res-Uim1)/dx) >=
                         -sb(i,j)*sqrt(1+(res-Umin)*(res-Umin)/(dy*dy))
                       )
                    ))
                {
                    res=(b-sqrt(delta))/MC(i,j);
```

```
            }
        }
    }
    return  res;
}


double rootOfK2(int i,int j,double Umin){

    double res;
    double b,c,delta;
    double Uip1=U(i+1,j);

    res=Uip1+dxl2(i,j);
    if (res > Umin){
        res=Umin+ME(i,j);
        if ( res > (Uip1-P(i,j)) ){
            b=MA(i,j)*Uip1+MB(i,j)*Umin+sa;
            c=MD(i,j)+MA(i,j)*Uip1*Uip1+MB(i,j)*Umin*Umin+sa2*Uip1;
            delta=b*b-MC(i,j)*c; if (delta<0) {delta=0;};
            res=(b+sqrt(delta))/MC(i,j);
            if (!(
                  (res>=Umin)&
                  (res>=(Uip1-sc))&
                  (
                   (-(res-Uip1)/dx) <=
                    -sb(i,j)*sqrt(1+(res-Umin)*(res-Umin)/(dy*dy))
                  )
                ))
            {
                res=(b-sqrt(delta))/MC(i,j);
            }
        }
    }
    return  res;
}
```

Roots in the case $I(i,j) \leq alpha$:

```
double firstPossibility(int i,int j,double Umin){
    double res=0;
    double x=U(i-1,j);
```

```
        double b,c,delta,t1,t2;
        b=MA(i,j)*x+MB(i,j)*Umin-sa;
        c=MD(i,j)+MA(i,j)*x*x+MB(i,j)*Umin*Umin-sa2*x;
        delta=b*b-MC(i,j)*c;
        if (delta<0) delta=0;
        if (MC(i,j)==0) { res=c/(2*b);}
        else{
            if (MC(i,j)>0) {
                t1=(b+sqrt(delta))/MC(i,j);
                t2=(b-sqrt(delta))/MC(i,j);
            }
            else {
                t1=(b-sqrt(delta))/MC(i,j);
                t2=(b+sqrt(delta))/MC(i,j);
            }
            if (t1 <= (x+sc)) res=t1;
            else res=t2;
        }
        return res;
    }
```

Initialisation of the variables:

```
    void initialiseAndComputeAllVar(void){

        // scalar variables:

        N=I.dimx(); M=I.dimy();
        dx=1./((double)I.dimx());
        dy=1./((double)I.dimy());
        gamma=sqrt(1-alpha*alpha);
        sa=alpha*gamma/dx;
        sa2=2*sa;
        sc=gamma*dx/alpha;
        sd=alpha*dx/gamma;

        // matrix variables:

        I2=I^2;
        I2ma2=I2-(alpha*alpha);
        MA=I2ma2/(dx*dx);
```

```
    MB=I2/(dy*dy);
    MC=MA+MB;
    MP=MA*MB;
    MD=I2-(gamma*gamma);
    ME0=(invImage(I2ma2))*(I2-1);
    ME0=sqrtI(oppImage(ME0));
    ME=ME0*dy;

    MBsa2=MB*sa2;
    sb=(invImage(sqrtI(I2ma2)))*alpha;
    sbdx=sb*dx;
    sbdxdy=sbdx*dy;

    IMAGE temp;
    temp=1-I2;
    temp=sqrtI(temp);temp=I*temp;
    l1= temp-alpha*gamma;
    l2= temp+alpha*gamma;
    temp=invImage(I2ma2);
    l1*=temp;
    l2*=temp;
    dxl1=l1*dx;
    dxl2=l2*dx;

    P=(invImage(I2ma2))*(-alpha*sqrt(1-alpha*alpha)*dx);
}
```

Bibliography

[1] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhauser, 1997.

[2] G. Barles. *Solutions de Viscosité des Equations de Hamilton–Jacobi.* Springer–Verlag, 1994.

[3] Guy Barles and P.E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, 4:271–283, 1991.

[4] M.J. Brooks, W. Chojnacki, and R. Kozera. Shading without shape. *Quarterly of Applied Mathematics*, L(1):27–38, 1992.

[5] A.R. Bruss. The eikonal equation: Some results applicable to computer vision. *Journal of Mathematical Physics*, 23(5):890–896, May 1982.

[6] M.G. Crandall. Viscosity solutions of Hamilton–Jacobi equations. In *Nonlinear Problems: Present and Future, Proc. 1st Los Alamos Conf., 1981*, volume 61, pages 117–125. North-Holland Math. Stud., 1982.

[7] M.G. Crandall, H. Ishii, and P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Soc.*, 27:1–67, 1992.

[8] M.G. Crandall and P.L. Lions. Viscosity solutions of Hamilton–Jacobi equations. *Trans. AMS*, 277:1–43, 1983.

[9] Paul Dupuis and John Oliensis. An optimal control formulation and related numerical methods for a problem in shape reconstruction. *The Annals of Applied Probability*, 4(2):287–346, 1994.

[10] J.-D. Durou and H. Maitre. On convergence in the methods of strat and smith for shape from shading. *The International Journal of Computer Vision*, 17(3):273–289, 1996.

[11] J.-D. Durou and D. Piau. Ambiguous shape from shading with critical points. *Journal of Mathematical Imaging and Vision*, 12:99–108, 2000.

[12] I. Ekeland and R. Temam. *Analyse Convexe et Problèmes Variationnels.* Etudes mathematiques. Dunod; Gauthier-Villars, Paris, Bruxelles, Montreal, 1974.

[13] M. Falcone and M. Sagona. An algorithm for the global solution of the shape-from-shading model. *International Conference on Image Analysis and Processing*, 1:596–603, 1997. LNCS 1310.

[14] M. Falcone, M. Sagona, and A. Seghini. A scheme for the shape-from-shading model with "black shadows". In *Proceedings of ENUMATH 2001*, 01.

[15] Maurizio Falcone and Charalampos Makridakis, editors. *numerical methods for viscosity solutions and applications*, volume 59 of *Series on advances in mathematics for applied sciences*. World Scientific, 2001.

[16] Berthold K. Horn and Michael J. Brooks, editors. *Shape from Shading*. The MIT Press, 1989.

[17] Berthold K.P. Horn. Height and Gradient from Shading. *The International Journal of Computer Vision*, 5(1):37–75, August 1990.

[18] B.K.P. Horn. Obtaining shape from shading information. In P.H. Winston, editor, *The Psychology of Computer Vision*. McGraw-Hill, New York, 1975.

[19] B.K.P Horn, R.S. Szeliski, and A.L. Yuille. Impossible shaded images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 15(2):166–169, 1993.

[20] K. Ikeuchi and B.K.P Horn. Numerical shape from shading and occluding boundaries. *Artificial Intelligence Journal*, 17:181–184, 1981.

[21] A. Ishii. A simple, direct proof of uniqueness for solutions of the hamilton-jacobi equations of eikonal type. *Proceedings of the American Mathematical Society*, pages 247–251, 1987.

[22] H. Ishii. Hamilton-jacobi equations with discontinuous hamiltonians on arbitrary open subsets. *Bull. Fac. Sci. Engrg. Chuo Univ.*, 28:33–77, 1985.

[23] H. Ishii. a boundary value problem of the dirichlet type for hamilton-jacobi equations. *Ann. Sci. Ecole Norm. Sup.*, 4(16):105–135, 1989.

[24] J. Kain and D.N. Ostrov. Numerical shape-from-shading for discontinuous photographic images. *The International Journal of Computer Vision*, 44(3):163–173, 2001.

[25] P.L. Lions. *Generalized Solutions of Hamilton–Jacobi Equations*. Number 69 in Research Notes in Mathematics. Pitman Advanced Publishing Program., 1982.

[26] P.L. Lions, E. Rouy, and A. Tourin. Shape-from-shading, viscosity solutions and edges. *Numer. Math.*, 64:323–353, 1993.

[27] D.N. Ostrov. Extending viscosity solutions to eikonal equations with discontinuous spatial dependence. *Nonlinear Anal.*, 42(4):709–736, 2000.

[28] E. Prados, O. Faugeras, and E. Rouy. Shape from shading and viscosity solutions. In *Proceedings of ECCV'02*, June 2002.

[29] E. Rouy and A. Tourin. A Viscosity Solutions Approach to Shape-from-Shading. *SIAM Journal of Numerical Analysis*, 29(3):867–884, June 1992.

[30] Ruo Zhang, Ping-Sing Tsai, James Edwin Cryer, and Mubarak Shah. Shape from shading: A survey. *IEEE Transactions On Pattern Analysis And Machine Intelligence*, 21(8):690–706, August 1999.

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